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Electromagnetic Propagation in Multimode Optical Fibers
Excited by Sources of Finite Bandwidth

FINAL REPORT

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The dependence of the temporal width of the impulse response on the length z of a two-mode optical fiber is examined. This quantity, which is proportional to z in the absence of mode coupling and to $z^{1/2}$ in the presence of weak random coupling among the guided modes, possesses a different dependence in the case of a deterministic resonant-coupling model, appropriate for describing a rather general class of actual situations. The relevant role played by the coherence time of the signal is demonstrated. (continued on reverse side)

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20. ABSTRACT (continued from other side)

The effect of strong mode coupling on modal dispersion in optical fibers has been investigated. The pulse dispersion turns out to be qualitatively different from the one relative to the weak-coupling case, while it exhibits a drastic reduction as compared with that of the uncoupled case. The role of the initial pulse length and of the source coherence time has been elucidated.

A coupled system of equations governing the propagation of a signal in a statistical ensemble of multimode optical fibers is presented. It describes, besides the usual average modal powers, the evolution of the interference terms between the mode amplitudes and of the modal power fluctuations. Our procedure allows us to treat the general nonstationary nonmonochromatic case of an arbitrary signal fed into the lightguide by a source possessing a finite spectral bandwidth. The introduction of modal power fluctuations permits us to establish a theorem connecting the value of the modal power, averaged over the fiber ensemble, with the actual one concerning a single fiber. These two values coincide, in the polychromatic case, for large values of the fiber length, thus providing the main result of the paper, that is the justification of the statistical approach to the problem of propagation. Furthermore, the analysis of the interference terms presents evidence for the difference between the propagation of an amplitude-modulated and a frequency-modulated signal.

The propagation in a multimode optical fiber of a finite bandwidth optical carrier modulated by a nonstationary signal is investigated. The fluctuations of the field due to random modecoupling are considered and the set of coupled equations describing their evolution is derived. In particular, this allows us to investigate the propagation of a frequency-modulated signal and to obtain a general theorem concerning the asymptotic behavior of mode-power fluctuations.

Under the present grant for the past three years we have been investigating the propagation of electromagnetic signals in optical fibers.

Our investigation has been quite fruitful. It has led to interesting results regarding finite-bandwidth propagation in multimode optical fibers, time-dependent propagation in multimode lightguides, modal dispersion in lightguides in the presence of strong coupling, and temporal spreading of a pulse propagating in a two-mode optical fiber.

Our results were circulated among the cognizant scientists of the U. S. Army Research Office and then published as a series of three papers in the Journal of the Optical Society of America and one paper in the AGARD Conference Proceedings, London Symposium, May, 1977.

In the following pages we have reproduced these four papers which describe in detail the work we have done under the present grant and constitute the body of our final report.

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Theory of time-dependent propagation in multimode lightguides*

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A coupled system of equations governing the propagation of a signal in a statistical ensemble of multimode optical fibers is presented. It describes, besides the usual average modal powers, the evolution of the interference terms between the mode amplitudes and of the modal power fluctuations. Our procedure allows us to treat the general nonstationary nonmonochromatic case of an arbitrary signal fed into the lightguide by a source possessing a finite spectral bandwidth. The introduction of modal power fluctuations permits us to establish a theorem connecting the value of the modal power, averaged over the fiber ensemble, with the actual one concerning a single fiber. These two values coincide, in the polychromatic case, for large values of the fiber length, thus providing the main result of the paper, that is the justification of the statistical approach to the problem of propagation. Furthermore, the analysis of the interference terms presents evidence for the difference between the propagation of an amplitude-modulated and a frequency-modulated signal.

I. INTRODUCTION

Multimode optical fibers with large-diameter core are particularly useful in connection with sources possessing intermediate degrees of spatial coherence, such as light-emitting diodes and laser diodes. These sources have a finite bandwidth $\delta\nu$, and, accordingly, the problem of propagation in the fiber cannot be simplified by assuming a monochromatic input. Indeed, a theory which takes into account the lack of monochromaticity (or the finite coherence time $t_c = 1/\delta\nu$) is necessary.

The description of the propagation of an electromagnetic signal in a multimode lightguide in the presence of mode coupling is a statistical one. This is a consequence of the fact that the guiding structure unavoidably presents some imperfections responsible for the coupling among the modes, whose randomness, together with the analytical complexity of the exact deterministic problem, leads in a natural way to the statistical approach. It consists of an introduction of an ensemble of macroscopically similar fibers, which differ among themselves only for the random behavior of the microscopic imperfections, and we consider the values of all the significant quantities averaged over this ensemble.[1] In this frame, the main emphasis has been placed on the evaluation of the average power carried by each propagating mode, both in the case of a monochromatic[1] and of a polychromatic signal.[2,3]

However, it is hardly necessary to remark, as always when dealing with a statistical approach, about the relevance of the fluctuations over the average value, since only their (relative) smallness assures that the average value of a given quantity can be safely assumed as the actual one. The fluctuations have been evaluated, only in the monochromatic case,[1,4] with a rather disappointing result. More precisely, it has been found that the normalized mean-square value of the power carried by each mode approaches 2 for long-traveled distances, so that there is a 100% uncertainty on the actual value of this quantity in a single fiber once the average value is known.

The above discussion gives evidence to the necessity of refining the theory of propagation in a multimode optical fiber in two ways. First, the existing treatment only deals with the average power per mode, which is connected with the square value of the corresponding mode amplitude, while a more complete description also requires the study of the evolution of the correlation terms between different mode amplitudes. In the course of the paper it will be shown that these are the only terms influenced by a frequency modulation of the input signal, an operation which, conversely, does not affect the behavior of the power per mode. Second, one needs a more complete investigation of the fluctuations, capable of covering the nonmonochromatic case. In fact, it will be proved that the finite bandwidth of the source has the positive consequence of reducing the magnitude of the fluctuations so that, for long-traveled distances, the average power per mode does coincide with the actual value in a single fiber.

The mentioned results are obtained in the frame of a general theory of nonstationary propagation, dealing with a carrier possessing a finite bandwidth and modulated in an arbitrary way, in which the complete description of the propagation and of the associated fluctuations is obtained in the form of a closed system of differential equations connecting all the correlation terms between mode amplitudes up to the fourth order. The present treatment generalizes that of Marcuse,[1] since it is applicable to the propagation of a polychromatic signal, a case which substantially differs from that relative to the monochromatic signal, as it is for example shown by the drastic change in the behavior of the fluctuations, which improves the predictability of the actual value of the single-mode power in the case of finite bandwidth.

II. PROPAGATION IN A SINGLE FIBER

The transverse part of a monochromatic electromagnetic field of angular frequency ω, propagating in the z direction along the axis of a (nearly) cylindrical dielectric fiber, can be expressed[1]

$$\mathbf{E}(\mathbf{r}, z, t) = \sum_m \mathbf{E}_m(\mathbf{r}, \omega) c_m(z, \omega)$$
$$\times \exp[i\omega t - i\beta_m(\omega)z], \qquad (1)$$

where $E_m(r, \omega)$ is the transverse electric field vector of the mth mode of the ideal cylindrical guiding structure, $\beta_m(\omega)$ is its relative propagation constant, and the c_m's are suitable expansion coefficients. In writing Eq. (1), we have assumed that the electric field can be expressed in terms of the forward-traveling guided modes alone; and we have allowed the expansion coefficients c_m to be z dependent in order to take into account the departure of the fiber shape from the ideal one to which the mode configurations E_m pertain. Here $r = (x, y)$ indicates the coordinates in a plane transverse to z and one needs to consider only the transverse part of the electric field $E(r, z, t)$ and of the magnetic field

$$H(r, z, t) = \sum_m H_m(r, \omega) c_m(z, \omega)$$
$$\times \exp[i\omega t - i\beta_m(\omega)z] \qquad (2)$$

for the purpose of evaluating the power carried by the field.

Equations (1) and (2) can be easily generalized to the nonmonochromatic case. In particular, if, as it is usually the case, the bandwidth $\delta\omega$ of the field fulfills the relation

$$\delta\omega/\omega_0 \ll 1 , \qquad (3)$$

where ω_0 is the central frequency, one can write

$$E(r, z, t) = \sum_m E_m(r, \omega_0) \int d\omega\, c_m(z, \omega)$$
$$\times \exp[i\omega t - i\beta_m(\omega)z] \qquad (4)$$

and an analogous expression for H. The complex Poynting vector is given by[5]

$$S = \tfrac{1}{2}\langle E \times H^* \rangle_{av} , \qquad (5)$$

where $\langle\ \rangle_{av}$ indicates the averaging operation over the fluctuations of the source. It is worth noting that the electromagnetic field possesses, in general, fast fluctuations associated with the carrier (that is with the source) and a slow variation associated with the superimposed signal. The averaging operation indicated as $\langle\ \rangle_{av}$ refers to the statistical ensemble pertinent to the source. Whenever the source is stationary, and the ergodic hypothesis is justified,[6] the quantity P_t in Eq. (6) represents the power averaged over a time of the order of $2\pi/\delta\omega$ and still retains the slow-time variation.

The power carried by the field through an arbitrary fiber section $z = $ const can be expressed

$$P_t = \text{Re} \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy\, S \cdot e_z , \qquad (6)$$

where e_z is the unit vector in the z direction and "Re" means "real part of." Taking advantage of the orthogonality relations between the guided modes[1]

$$\int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy\, e_z \cdot E_n(r, \omega_0) \times H_m^*(r, \omega_0) = 2P\delta_{mn} , \qquad (7)$$

where P is a positive normalization constant and δ_{mn} is Kronecker's symbol, one obtains through Eqs. (4)–(7),

$$P_t = P \sum_m \int d\omega \int d\omega' \langle c_m(z, \omega) c_m^*(z, \omega') \rangle_{av}$$

$$\times \exp\{i(\omega - \omega')t - i[\beta_m(\omega) - \beta_m(\omega')]z\} = \sum_m P_m(z, t) , \qquad (8)$$

where $P_m(z, t)$ is a positive quantity, as it is immediately verified by inspection, which can be interpreted as the electromagnetic power carried by the mth mode.

If mode coupling is absent, one has

$$c_m(z, \omega) = c_m(0, \omega) = \frac{1}{4\pi P} \int_{-\infty}^{+\infty} dt\, e^{-i\omega t}$$
$$\times \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy\, e_z \cdot E(r, 0, t) \times H_m^*(r, \omega_0)$$
$$= \frac{1}{4\pi P} \int_{-\infty}^{+\infty} dt\, e^{-i\omega t} e^{i\omega_0 t} F(t)$$
$$\times \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy\, e_z \cdot G(r) \times H_m^*(r, \omega_0) , \qquad (9)$$

having assumed, for the sake of simplicity, the electric field at $z = 0$ to be of the form

$$E(r, 0, t) = e^{i\omega_0 t} F(t) G(r) . \qquad (10)$$

Taking advantage of Eq. (3), one can approximately write

$$\beta_m(\omega) = \beta_m(\omega_0) + (\omega - \omega_0)/V_m , \qquad (11)$$

with

$$V_m^{-1} = [d\beta_m(\omega)/d\omega]_{\omega = \omega_0} , \qquad (12)$$

and Eq. (8) yields

$$P_t = \frac{1}{4P} \sum_m \left\langle \left| F\left(t - \frac{z}{V_m}\right) \right|^2 \right\rangle_{av}$$
$$\times \left| \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy\, G(r) \times H_m^*(r, \omega_0) \cdot e_z \right|^2 . \qquad (13)$$

Equation (13) expresses the fact that the modulation imposed at $z = 0$ travels in each mode with the mode characteristic group velocity V_m; in particular, that frequency modulation, which affects the phase of $F(t)$, is ineffective in influencing the total power carried through a fiber section.

In order to satisfactorily treat the situation in which mode coupling is present, one has to resort to an ensemble-average approach. More precisely, one introduces a statistical ensemble of macroscopically equivalent fibers possessing microscopic random imperfections and then considers the ensemble average, indicated by the symbol $\langle\cdots\rangle$. Accordingly, all the significant quantities will be expressed by means of two statistically independent averaging operations, the first of which, indicated by $\langle\ \rangle_{av}$, refers to the source, while the second, indicated by angular brackets, is associated with the introduction of the statistical fiber ensemble. Thus one has, for example, the total average power carried through a fiber section given by

$$\langle P_t \rangle = \sum_m \langle P_m(z, t) \rangle$$
$$= P \sum_m \int d\omega \int d\omega' \langle\langle c_m(z, \omega) c_m^*(z, \omega') \rangle_{av}\rangle$$
$$\times \exp\{i(\omega - \omega')t - i[\beta_m(\omega) - \beta_m(\omega')]z\} . \qquad (14)$$

III. STATISTICAL THEORY OF TIME-DEPENDENT PROPAGATION IN AN ENSEMBLE OF FIBERS

In order to study the space-time evolution of $\langle P_m(z, t)\rangle$ in the fiber, one has to, according to Eq. (14), investigate the behavior of the quantity $\langle\langle c_m(z, \omega)c_m^*(z, \omega')\rangle\rangle_{av}$. The starting point is the set of equations describing the deterministic evolution of the c_m's in a single fiber. They read[1]

$$\frac{dc_m(z, \omega)}{dz} = \sum_j A_{mj}(z, \omega, \omega_0)c_j(z, \omega), \quad m = 1, 2, \ldots, \tag{15}$$

where

$$A_{mj}(z, \omega, \omega_0) = K_{mj}(z, \omega_0) \times \exp\{i[\beta_m(\omega) - \beta_j(\omega)]z\}, \tag{16}$$

the K_{mj}'s being suitable coupling coefficients obeying the relation

$$K_{mj} = -K_{jm}^*. \tag{17}$$

An equation completely analogous to Eq. (15) holds obviously true for ω' and reads

$$\frac{dc_m(z, \omega')}{dz} = \sum_j A_{mj}(z, \omega', \omega_0)c_j(z, \omega'), \quad m = 1, 2, \ldots. \tag{18}$$

From Eqs. (15) and (18) one can immediately derive the following set of equations:

$$\frac{d}{dz}[c_m^*(z, \omega')c_j(z, \omega)]$$
$$= \sum_k \{A_{jk}(z, \omega, \omega_0)c_m^*(z, \omega')c_k(z, \omega) \tag{19}$$
$$+ A_{mk}^*(z, \omega', \omega_0)c_j(z, \omega)c_k^*(z, \omega')\}, \quad m, j = 1, 2, \ldots.$$

At this point, one can follow a procedure analogous to that introduced in the monochromatic case ($\omega = \omega'$) (Ref. 4); that is, one formally integrates Eq. (19) and then substitutes in the resulting expression for $c_m^*(z, \omega')c_j(z, \omega)$ the formal solution for the quantities $c_m^*(z, \omega')c_k(z, \omega)$ and $c_j(z, \omega)c_k^*(z, \omega')$. After doing this, one can perform the double-averaging operation on the resulting equations.

If the coupling process is stationary to order two in the z direction, that is,

$$\langle K_{mj}(z, \omega_0)K_{rs}(z', \omega_0)\rangle = F_{mjrs}(|z - z'|), \tag{20}$$

and if

$$\langle K_{mj}(z, \omega_0)\rangle = 0, \tag{21}$$

by assuming that the mode-field amplitudes do not appreciably vary over distances comparable with the typical correlation length D of the correlation function defined in Eq. (20), one obtains

$$\frac{d}{dz}X_{mr}(z, \omega, \omega') = \sum_j \sum_k X_{kr}(z, \omega, \omega')\exp\{i[\beta_m(\omega) - \beta_k(\omega)]z\}\int_0^\infty \langle K_{mj}(z')K_{jk}(0)\rangle$$

$$\times \exp\{-i[\beta_j(\omega) - \beta_k(\omega)]z'\}dz' + X_{jk}(z, \omega, \omega')\exp\{i[\beta_m(\omega) - \beta_j(\omega) + \beta_k(\omega') - \beta_r(\omega')]z\}$$

$$\times \int_0^\infty \langle K_{mj}(z')K_{rk}^*(0)\rangle\exp\{-i[\beta_k(\omega') - \beta_r(\omega')]z'\}dz' + X_{kj}(z, \omega, \omega')$$

$$\times \exp\{i[\beta_j(\omega') - \beta_r(\omega') + \beta_m(\omega) - \beta_k(\omega)]z\}\int_0^\infty \langle K_{rj}^*(z')K_{mk}(0)\rangle\exp\{-i[\beta_m(\omega) - \beta_k(\omega)]z'\}dz'$$

$$+ X_{mk}(z, \omega, \omega')\exp\{i[\beta_k(\omega') - \beta_r(\omega')]z\}\int_0^\infty \langle K_{rj}^*(z')K_{jk}^*(0)\rangle\exp\{-i[\beta_k(\omega') - \beta_j(\omega')]z'\}dz', \tag{22}$$

where we have defined

$$X_{mr}(z, \omega, \omega') = \langle\langle c_m(z, \omega)c_r^*(z, \omega')\rangle_{av}\rangle; \tag{23}$$

and the dependence on ω_0 of the coupling coefficients has been omitted.

It can now be observed that only nonoscillatory exponential terms in z contribute appreciably to the right-hand side of Eq. (22), since integrals over z of rapidly oscillating terms can be neglected with respect to integrals of slowly varying functions (rotating-wave approximation).[7] More precisely, in order to benefit from this approximation, one has to assume that the c_m's do not undergo relevant variations over distances of the kind $[\beta_m(\omega) - \beta_k(\omega)]^{-1}$ and $[\beta_m(\omega) - \beta_j(\omega) + \beta_r(\omega) - \beta_s(\omega)]^{-1}$, unless, respectively, $m = k$ and $m = j$, $r = s$ or $m = s$, $j = r$. Under the same hypothesis, it follows that the c_m's do not vary over the distance $[\beta_m(\omega) - \beta_j(\omega) + \beta_r(\omega') - \beta_s(\omega')]^{-1}$, unless $m = j$, $r = s$ or $m = s$, $j = r$, since

$$\beta_m(\omega) - \beta_j(\omega) + \beta_r(\omega') - \beta_s(\omega') \simeq \beta_m(\omega) - \beta_j(\omega) + \beta_r(\omega) - \beta_s(\omega), \tag{24}$$

provided that the condition $|\omega - \omega'|/\omega \ll 1$ is fulfilled [see Eq. (3)]. Equation (24) can be justified by observing that

$$\beta_m(\omega) - \beta_j(\omega) + \beta_r(\omega') - \beta_s(\omega')$$
$$\simeq \beta_m(\omega) - \beta_j(\omega) + \beta_r(\omega) - \beta_s(\omega) + \left(\frac{d\beta_r}{d\omega} - \frac{d\beta_s}{d\omega}\right)(\omega' - \omega), \tag{25}$$

and that

$$\left|\frac{(d\beta_r/d\omega - d\beta_s/d\omega)(\omega' - \omega)}{\omega}\right|$$
$$\ll \frac{|\beta_m(\omega) - \beta_j(\omega) + \beta_r(\omega) - \beta_s(\omega)|}{\omega}, \tag{26}$$

Eq. (26) being always valid if the differences between the inverses of the group velocities are of the same order of the differences between the inverses of the phase velocities.

By taking advantage of the rotating-wave approximation, Eq. (22) yields

$$\frac{d}{dz} X_{mr}(z,\ \omega,\ \omega') = X_{mr}(z,\ \omega,\ \omega')$$

$$\times \left(\sum_j \int_0^\infty \langle K_{mj}(z') K_{jm}(0) \rangle \exp\{- i[\beta_j(\omega) - \beta_m(\omega)] z'\} dz' \right.$$

$$+ \sum_j \int_0^\infty \langle K_{rj}^*(z') K_{jr}^*(0) \rangle \exp\{- i[\beta_r(\omega') - \beta_j(\omega')] z'\} dz'$$

$$\left. + \int_{-\infty}^{+\infty} \langle K_{mm}(z') K_{rr}^*(0) \rangle dz' \right), \quad m \neq r. \tag{27}$$

It is not difficult to show (see Appendix) that the co-efficient of X_{mr} on the right-hand side of Eq. (27) has a nonpositive real part, so that Eq. (27) can be rewritten

$$\frac{d}{dz} X_{mr}(z,\ \omega,\ \omega') = - g_{mr} X_{mr}(z,\ \omega,\ \omega'), \quad m \neq r, \tag{28}$$

with $\mathrm{Re} g_{mr} \geq 0$, whose solution is

$$X_{mr}(z,\ \omega,\ \omega') = \langle c_m(0,\ \omega) c_r^*(0,\ \omega') \rangle_{av} e^{-g_{mr} z}, \quad m \neq r, \tag{29}$$

where $c_m(0,\ \omega) \equiv c_m(z = 0,\ \omega)$.

Equation (29) shows that the cross-correlation terms in the mode-field amplitudes practically vanish after a distance of the order $1/\mathrm{Re} g_{mr}$; or that they remain identically zero if they possess this value at $z = 0$ (excitation by a spatially incoherent source).

For the diagonal terms $m = r$, we obtain from Eq. (22),

$$\frac{d}{dz} X_{mm}(z,\ \omega,\ \omega') = - \left(\sum_{j \neq m} h_{mj} \right) X_{mm}(z,\ \omega,\ \omega')$$

$$+ \sum_{j \neq m} h_{mj} \exp\{i [\beta_m(\omega) - \beta_m(\omega')] z$$

$$+ i [\beta_j(\omega') - \beta_j(\omega)] z\} X_{jj}(z,\ \omega,\ \omega'), \tag{30}$$

where

$$h_{mj} = \int_{-\infty}^{+\infty} \langle K_{mj}(z') K_{mj}^*(0) \rangle$$

$$\times \exp\{i [\beta_j(\omega_0) - \beta_m(\omega_0)] z'\} dz' \tag{31}$$

is a real non-negative quantity (see Appendix). In deriving Eq. (30) we have taken advantage of the fact that the value of h_{mj} is not substantially modified by the substitution $\omega - \omega_0$ and $\omega' - \omega_0$ performed in writing Eq. (31).

The set of Eqs. (30) describes in full generality the evolution of the autocorrelation functions of the mode-field amplitudes and allows in principle the evaluation of $\langle P_t(z,\ t) \rangle$ by means of Eq. (14). It is, however, more convenient to deal directly with a set of equations connecting the average powers $\langle P_m(z,\ t) \rangle$ carried by each mode. By recalling the expression of $\langle P_m(z,\ t) \rangle$ given by Eq. (14), one obtains, with the help of Eq. (30),

$$\frac{\partial}{\partial z} \langle P_m(z,\ t) \rangle = - \left(\sum_{j \neq m} h_{mj} \right) \langle P_m(z,\ t) \rangle$$

$$+ \sum_{j \neq m} h_{mj} \langle P_j(z,\ t) \rangle + P \int d\omega \int d\omega' \, i[\beta_m(\omega') - \beta_m(\omega)]$$

$$\times \exp[i(\omega - \omega')t] \exp\{- i[\beta_m(\omega) - \beta_m(\omega')] z\} X_{mm}(z,\ \omega,\ \omega'). \tag{32}$$

The last term on the right-hand side of Eq. (32) can

be rewritten, with the help of Eq. (11),

$$- \frac{1}{V_m} \frac{\partial}{\partial t} \langle P_m(z,\ t) \rangle, \tag{33}$$

so that finally one has

$$\frac{\partial}{\partial z} \langle P_m(z,\ t) \rangle + \frac{1}{V_m} \frac{\partial}{\partial t} \langle P_m(z,\ t) \rangle$$

$$= \sum_j h_{mj} [\langle P_j(z,\ t) \rangle - \langle P_m(z,\ t) \rangle], \tag{34}$$

which constitutes the set of coupled equations describing the propagation of a pulse in a multimode lightguide. At this stage, distributed losses can be easily incorporated by adding the term $- 2\alpha_m \langle P_m(z,\ t) \rangle$ on the right-hand side of Eq. (34).

The set of Eqs. (34), which describes the time-dependent propagation of the average modal powers, is identical with the one worked out by previous authors.[1-3] The present derivation has the advantage of clearly showing that the limits of validity of Eq. (34) rely on the small-bandwidth assumption expressed by Eq. (3), as well as on the hypothesis of slow variation of the field amplitudes required also in the stationary case.[1,4]

IV. PROPAGATION OF A FREQUENCY-MODULATED SIGNAL

The form of the set of Eqs. (34) implies that the average power $\langle P_m(z,\ t) \rangle$ carried by each mode is not influenced by the frequency modulation of the input signal; that is, that this kind of modulation cannot be transmitted by the fiber, as long as one collects all the light coming out from the whole fiber section. This circumstance is no longer true if one considers the power transmitted through a smaller area σ of the section. In this case, the orthogonality of the modes is no longer effective in canceling the cross-correlation terms between different modes, and the expression of the power $\langle P_t^\sigma \rangle$ carried through this area contains also nondiagonal contributions. More precisely, one has, according to Eq. (5),

$$\langle P_t^\sigma \rangle = \frac{1}{2} \mathrm{Re} \sum_m \sum_r T_{mr}, \tag{35}$$

with

$$T_{mr}(z,\ t) = \iint_\sigma dx\, dy\, \mathbf{E}_m(\mathbf{r},\ \omega_0) \times \mathbf{H}_r^*(\mathbf{r},\ \omega_0) \cdot \mathbf{e}_s$$

$$\times \int d\omega \int d\omega' \exp\{i(\omega - \omega')t$$

$$- i [\beta_m(\omega) - \beta_r(\omega')] z\} X_{mr}(z,\ \omega,\ \omega'). \tag{36}$$

Recalling Eq. (29), one can rewrite the nondiagonal contributions as

$$T_{mr}(z,\ t) = \exp(- g_{mr} z) \iint_\sigma dx\, dy\, \mathbf{E}_m(\mathbf{r},\ \omega_0) \times \mathbf{H}_r^*(\mathbf{r},\ \omega_0) \cdot \mathbf{e}_s$$

$$\times \int d\omega \int d\omega' \exp\{i(\omega - \omega')t - i [\beta_m(\omega) - \beta_r(\omega')] z\}$$

$$\times \langle c_m(0,\ \omega) c_r^*(0,\ \omega') \rangle_{av}, \quad m \neq r, \tag{37}$$

which shows that the T_{mr}'s $(m \neq r)$ evolve as in the absence of mode coupling, except for the fact that their envelope undergoes an exponential decay with the fiber length z. In particular, if the source is spatially incoherent, then $T_{mr}(z, t) = 0$ $(m \neq r)$, and $\langle P_i^c \rangle$ is not influenced by the modulation of the signal. If, conversely, the source possesses some degree of spatial coherence, then the T_{mr}'s $(m \neq r)$ are sensibly different from zero over a certain fiber length, and Eqs. (9)–(12) can be recalled for determining their evolution, thus getting

$$T_{mr}(z,\ t) \propto e^{-g_{mr}z} \left\langle F\left(t - \frac{z}{V_m}\right) F^*\left(t - \frac{z}{V_r}\right) \right\rangle_{av}, \quad m \neq r. \tag{38}$$

If one assumes the exciting source to be a frequency-modulated single-mode laser, then

$$F(t) = \exp\left[i\phi(t) + i\gamma t^2 \right], \tag{39}$$

where $\phi(t)$ is a normally distributed random function responsible for the laser linewidth, γ is a real constant proportional to the frequency-modulation rate, and

$$T_{mr}(z,\ t) \propto \exp(-g_{mr}z)\exp\left(-\frac{|1/V_m - 1/V_r|z}{t_c}\right)$$
$$\times \exp\left[i\gamma z^2 \left(\frac{1}{V_m^2} - \frac{1}{V_r^2}\right) + 2i\gamma t\left(\frac{z}{V_r} - \frac{z}{V_m}\right)\right], \quad m \neq r, \tag{40}$$

$t_c = 1/\delta\nu$ being the coherence time of the source. Equation (40) shows that T_{mr} goes to zero over the shortest of the two distances $1/g_{mr}$ and $t_c|1/V_m - 1/V_r|$, the latter being the traveled length over which two modes m and r started from $z = 0$ at the same instant acquire

a time delay larger than t_c.[8] On the fiber length over which the T_{mr}'s $(m \neq r)$ are still different from zero, their time dependence, which is unaffected by mode coupling, determines the time behavior of $\langle P_i^c \rangle$, since the diagonal terms T_{mm}, being proportional to $\langle P_m \rangle$, do not depend on time.

V. BEHAVIOR OF FLUCTUATIONS IN THE NONSTATIONARY CASE

In order to deal with practical situations, in which a single fiber is usually employed, it is desirable to evaluate the fluctuations of the relevant quantities around their average values. In the monochromatic case, this has been done[1,4] by calculating for a lossless fiber the asymptotic value (in z) of the normalized variance

$$(\langle P_m^2 \rangle - \langle P_m \rangle^2)/\langle P_m \rangle^2. \tag{41}$$

For large z, the resulting value is 1, which implies a 100% uncertainty on the actual value of the mth mode power in a single fiber. This rather disappointing situation drastically changes in the case of a stationary signal possessing a finite bandwidth $\delta\omega$, where it has been shown that the asymptotic normalized variance tends to vanish.[9] In this section, we study the evolution of the fourth-order field-amplitude correlation functions relative to different modes and different frequencies, which will allow us to generalize the result of Ref. 9 to nonstationary situations.

The starting point is the set of Eqs. (19), which immediately furnishes

$$\frac{d}{dz} \langle c_m(z, \omega)c_r^*(z, \omega')\rangle_{av}\langle c_s(z, \omega'')c_n^*(z, \omega''')\rangle_{av}$$

$$= \sum_j \{ A_{mj}(z, \omega)\langle c_j(z, \omega)c_r^*(z, \omega')\rangle_{av}\langle c_s(z, \omega'')c_n^*(z, \omega''')\rangle_{av} + A_{rj}^*(z, \omega')\langle c_m(z, \omega)c_j^*(z, \omega')\rangle_{av}\langle c_s(z, \omega'')c_n^*(z, \omega''')\rangle_{av}$$

$$+ A_{sj}(z, \omega'')\langle c_m(z, \omega)c_r^*(z, \omega')\rangle_{av}\langle c_j(z, \omega'')c_n^*(z, \omega''')\rangle_{av} + A_{nj}^*(z, \omega'')\langle c_m(z, \omega)c_r^*(z, \omega')\rangle_{av}\langle c_s(z, \omega'')c_j^*(z, \omega''')\rangle_{av} \}. \tag{42}$$

It is worthwhile to emphasize at this point that the source-average $\langle\ \rangle_{av}$ in Eq. (42) operates on mode-amplitude pairs successively. In fact, the basic quantity measured by the detector in a single fiber is the modal power $P_m(z, t)$ [see Eq. (8)], which contains the second-order average $\langle c_m(z, \omega)c_m^*(z, \omega')\rangle_{av}$, so that the evolution of ensemble averages of the kind $\langle P_m(z, t)P_n(z, t)\rangle$ involves the fourth-order products $\langle c_m(z, \omega)c_m^*(z, \omega')\rangle_{av}\langle c_n(z, \omega'')c_n^*(z, \omega''')\rangle_{av}$. In turn, the evolution of $\langle c_m(z, \omega)c_m^*(z, \omega')\rangle_{av}$ is connected with nondiagonal second-order products of the type $\langle c_j(z, \omega)c_r^*(z, \omega')\rangle_{av}$, which justifies the appearance of $\langle\ \rangle_{av}\langle\ \rangle_{av}$ in Eq. (42).

By taking $r = m$, $s = n$ and $\omega = \omega'$, $\omega'' = \omega'''$, it is not difficult to obtain from Eq. (42), by means of a direct extension of the procedure outlines in Sec. III, the following set of equations:

$$\frac{d}{dz}\langle \tilde{P}_m(z, \omega)\tilde{P}_n(z, \omega'')\rangle = -\left(\sum_j h_{mj}\right)\langle \tilde{P}_m(z, \omega)\tilde{P}_n(z, \omega'')\rangle + \sum_j h_{mj}\langle \tilde{P}_j(z, \omega)\tilde{P}_n(z, \omega'')\rangle$$

$$+ \Big(\langle\langle c_n^*(z, \omega'')c_m(z, \omega'')\rangle_{av}\langle c_n(z, \omega)c_m^*(z, \omega)\rangle_{av}\rangle \exp\{i[\beta_m(\omega) - \beta_m(\omega'') + \beta_n(\omega'') - \beta_n(\omega)]z\}(1 - \delta_{mn})\int_0^\infty \langle K_{mn}(z')K_{mn}(0)\rangle$$

$$\times \exp\{-i[\beta_n(\omega'') - \beta_m(\omega'')]z'\}dz' + \text{c.c}\Big) + \Big(\sum_{j \neq m}\langle\langle c_j(z, \omega)c_m^*(z, \omega)\rangle_{av}\langle c_m(z, \omega'')c_j^*(z, \omega'')\rangle_{av}\rangle$$

$$\times \exp\{i[\beta_m(\omega) - \beta_m(\omega'') + \beta_s(\omega'') - \beta_j(\omega)]z\}\delta_{rm}\int_0^\infty \langle K_{mj}(z')K_{mj}^*(0)\rangle \exp\{-i[\beta_j(\omega'') - \beta_m(\omega'')]z'\}dz' + \text{c.c}\Big) + N_{mn}, \tag{43}$$

where N_{ex} represents same terms with the exchange $m \to n$, $\omega \to \omega''$, having defined

$$\tilde{P}_m(z, \omega) = \langle |c_m(z, \omega)|^2 \rangle_{ex} . \tag{44}$$

While the set of Eqs. (43) possesses a general validity under the usual hypotheses justyfying the rotating-wave approximation, it can be cast in a particularly simple form if the relation

$$\left| \left(\frac{d\beta_m}{d\omega} - \frac{d\beta_j}{d\omega} \right)(\omega - \omega'') \right| z \gg 2\pi , \quad j \neq m \tag{45}$$

is fulfilled and if the field amplitudes do not appreciably vary over a distance $z_c^{(m,j)}$ given by [compare with Eq. (40)]

$$z_c^{(m,j)} = |(1/V_m - 1/V_j)|^{-1} 2\pi/\delta\omega . \tag{46}$$

In fact, in this case, one can again take advantage of the rotating-wave approximation on the right-hand side of Eq. (43), which reduces to

$$\frac{d}{dz} \langle \tilde{P}_m(z, \omega) \tilde{P}_n(z, \omega'') \rangle$$

$$= \sum_j h_{mj} [\langle \tilde{P}_j(z, \omega) \tilde{P}_n(z, \omega'') \rangle - \langle \tilde{P}_m(z, \omega) \tilde{P}_n(z, \omega'') \rangle]$$

$$+ \sum_j h_{nj} [\langle \tilde{P}_j(z, \omega'') \tilde{P}_m(z, \omega) \rangle - \langle \tilde{P}_n(z, \omega'') \tilde{P}_m(z, \omega) \rangle] . \tag{47}$$

From Eq. (47), in the case $m = n$ and for values of z large enough so its left-hand side vanishes, it is immediate to deduce the following asymptotic equality:

$$\langle \tilde{P}_j(z, \omega) \tilde{P}_m(z, \omega'') \rangle_a + \langle \tilde{P}_j(z, \omega'') \tilde{P}_m(z, \omega) \rangle_a$$

$$= 2 \langle \tilde{P}_m(z, \omega) \tilde{P}_m(z, \omega'') \rangle_a , \tag{48}$$

where the subscript a stands for asymptotic. On the other hand, energy conservation along the fiber requires that

$$\sum_j \tilde{P}_j(z, \omega) = \sum_j \tilde{P}_j(0, \omega) \quad \text{for every } \omega, \tag{49}$$

which in turn implies

$$\sum_m \langle \tilde{P}_m(z, \omega) \tilde{P}_m(z, \omega'') \rangle + \frac{1}{2} \sum_{m \neq j} \sum \langle \tilde{P}_j(z, \omega) \tilde{P}_m(z, \omega'') \rangle$$

$$+ \frac{1}{2} \sum_{m \neq j} \sum \langle \tilde{P}_j(z, \omega'') \tilde{P}_m(z, \omega) \rangle$$

$$= \sum_m \tilde{P}_m(0, \omega) \sum_j \tilde{P}_j(0, \omega'') , \tag{50}$$

a relation which asymptotically yields, with the help of Eq. (48),

$$\sum_m \langle \tilde{P}_m(z, \omega) \tilde{P}_m(z, \omega'') \rangle_a + N(N-1) \langle \tilde{P}_m(z, \omega) \tilde{P}_m(z, \omega'') \rangle_a$$

$$= \sum_m \tilde{P}_m(0, \omega) \sum_j \tilde{P}_j(0, \omega'') , \tag{51}$$

where N is the total number of guided modes. Recalling now the asymptotic equipartition condition, expressing the fact that, in a lossless case, the power per mode is the same in every mode for z sufficiently large, [1]

$$\langle \tilde{P}_m(z, \omega) \rangle_a = \frac{1}{N} \sum_j \tilde{P}_j(0, \omega) \quad \text{for every } \omega , \tag{52}$$

Eq. (51) can be rewritten

$$\sum_m \langle \tilde{P}_m(z, \omega) \tilde{P}_m(z, \omega'') \rangle_a + N(N-1) \langle \tilde{P}_m(z, \omega) \tilde{P}_m(z, \omega'') \rangle_a$$

$$= N^2 \langle \tilde{P}_m(z, \omega) \rangle_a \langle \tilde{P}_m(z, \omega'') \rangle_a , \tag{53}$$

from which follows the relevant expression

$$\langle \tilde{P}_m(z, \omega) \tilde{P}_m(z, \omega'') \rangle_a = \langle \tilde{P}_m(z, \omega) \rangle_a \langle \tilde{P}_m(z, \omega'') \rangle_a . \tag{54}$$

We now observe that the energy $I_m(z)$ pertaining to the mth mode

$$I_m(z) = \int_{-\infty}^{\infty} P_m(z, t) dt , \tag{55}$$

can be expressed by means of Eqs. (8) and (44) as

$$I_m(z) = 2\pi P \int \tilde{P}_m(z, \omega) d\omega . \tag{56}$$

The normalized variance of the statistical quantity I_m can be accordingly written

$$\frac{\langle I_m^2 \rangle - \langle I_m \rangle^2}{\langle I_m \rangle^2} = \left(\int d\omega \int d\omega'' \langle \tilde{P}_m(z, \omega) \tilde{P}_m(z, \omega'') \rangle \right.$$

$$\left. - \int d\omega \int d\omega'' \langle \tilde{P}_m(z, \omega) \rangle \langle \tilde{P}_m(z, \omega'') \rangle \right)$$

$$\times \left(\int d\omega \int d\omega'' \langle \tilde{P}_m(z, \omega) \rangle \langle \tilde{P}_m(z, \omega'') \rangle \right)^{-1} . \tag{57}$$

Since, for $z \gg z_c^{(m,j)}$, most couples of frequencies in the relevant integration domain fulfill Eq. (45), Eq. (54) applies and, consequently, the normalized variance tends asymptotically to vanish, that is

$$(\langle I_m^2 \rangle_a - \langle I_m \rangle_a^2)/\langle I_m \rangle_a^2 = 0 . \tag{58}$$

This implies that no statistical uncertainty is present in I_m, so that equipartition of the total energy among the various modes takes place in the single fiber. As a particular case of this general statement, the asymptotic power P_m has the same property in the stationary regime, as it has been shown elsewhere, [9] since in this situation I_m and P_m are proportional.

The complete investigation of the behavior of fluctuations in the nonstationary regime requires the analysis of the quantities $\langle P_m^2(z, t) \rangle$ and $\langle P_m(z, t) P_n(z, t) \rangle$. To this end, the system of Eqs. (42) has to be specified to the case $r = m$, $s = n$, since

$$\langle P_m(z, t) P_n(z, t) \rangle = P^2 \int d\omega \int d\omega' \int d\omega'' \int d\omega''' \exp[i(\omega - \omega')t] \exp\{i(\omega'' - \omega''')t - i[\beta_m(\omega) - \beta_m(\omega')]z$$

$$- i[\beta_n(\omega'') - \beta_n(\omega''')]z\} \langle\langle c_m(z, \omega) c_m^*(z, \omega') \rangle_m \langle c_n(z, \omega'') c_n^*(z, \omega''') \rangle_m \rangle . \tag{59}$$

By means of the general procedure followed in this paper, it is possible to deduce a set of equations governing the evolution of $\langle\langle c_m(z, \omega) c_m^*(z, \omega') \rangle_a \langle c_n(z, \omega'') c_n^*(z, \omega''') \rangle_m \rangle$. In order to simplify the notation, let us define

$$Y_{mr}(z, \omega, \omega') = \exp[i\beta_r(\omega')z - i\beta_m(\omega)z]\langle c_m(z, \omega)c_r^*(z, \omega')\rangle_{av} . \tag{60}$$

The final form of the closed system of equations then reads

$$\frac{d}{dz}\langle Y_{mn}(z, \omega, \omega')Y_{nn}(z, \omega'', \omega''')\rangle = i[\beta_m(\omega') - \beta_m(\omega) + \beta_n(\omega''') - \beta_n(\omega'')]\langle Y_{mm}(z, \omega, \omega')Y_{nn}(z, \omega'', \omega''')\rangle$$

$$- \left[\sum_h (h_{hm} + h_{hn})\right]\langle Y_{mm}(z, \omega, \omega')Y_{nn}(z, \omega'', \omega''')\rangle + \sum_h h_{hn}\langle Y_{mm}(z, \omega, \omega')Y_{hh}(z, \omega'', \omega''')\rangle$$

$$+ \sum_h h_{hm}\langle Y_{hh}(z, \omega, \omega')Y_{nn}(z, \omega'', \omega''')\rangle - (1 - \delta_{mn})h_{mn}[\langle Y_{nm}(z, \omega, \omega')Y_{mn}(z, \omega'', \omega''')\rangle$$

$$+ \langle Y_{mn}(z, \omega, \omega')Y_{nm}(z, \omega'', \omega''')\rangle] + \delta_{mn}\left(\sum_{h\neq m} h_{mh}[\langle Y_{hm}(z, \omega, \omega')Y_{mh}(z, \omega'', \omega''')\rangle + \langle Y_{mh}(z, \omega, \omega')Y_{hm}(z, \omega'', \omega''')\rangle]\right) \tag{61}$$

and

$$\frac{d}{dz}\langle Y_{mr}(z, \omega, \omega')Y_{rm}(z, \omega'', \omega''')\rangle = i[\beta_r(\omega') - \beta_m(\omega) + \beta_m(\omega''') - \beta_r(\omega'')]\langle Y_{mr}(z, \omega, \omega')Y_{rm}(z, \omega'', \omega''')\rangle$$

$$- \left(\sum_{h\neq r} h_{hr} + \sum_{h\neq m} h_{hm}\right)\langle Y_{mr}(z, \omega, \omega')Y_{rm}(z, \omega'', \omega''')\rangle + \sum_{h\neq r} h_{rh}\langle Y_{mh}(z, \omega, \omega')Y_{hm}(z, \omega'', \omega''')\rangle + \sum_{h\neq m} h_{mh}\langle Y_{hr}(z, \omega, \omega')$$

$$\times Y_{rh}(z, \omega'', \omega''')\rangle - h_{mr}[\langle Y_{rr}(z, \omega, \omega')Y_{mm}(z, \omega'', \omega''')\rangle + \langle Y_{mm}(z, \omega, \omega')Y_{rr}(z, \omega'', \omega''')\rangle] , \quad m \neq r . \tag{62}$$

The set of Eqs. (61) and (62) represents, together with the set of Eqs. (30), a fairly complete analytical description of nonstationary propagation. In fact, while Eqs. (30) concern the evolution of the average power per mode, Eqs. (61) and (62) allow one to describe the fluctuations around this quantity in terms of fourth-order averages. One can observe that, in the general case, it is not possible to obtain a system of equations containing only the quantities $\langle P_m P_n\rangle$, while this is possible in the monochromatic stationary propagation.[4] This is related to the essential role of the average over the source, which does not allow the nondiagonal terms $\langle Y_{mr}(\omega, \omega')$ $\times Y_{rm}(\omega'', \omega''')\rangle$ to be rearranged as products $\langle Y_{mm}(\omega, \omega''')Y_{rr}(\omega'', \omega')\rangle$.

VI. CONCLUSIONS

We have derived a set of statistical coupled equations describing the time-dependent propagation of a finite-bandwidth carrier, which can be either amplitude or frequency modulated, placing into evidence the limits of validity of the approach, which involve both the value of the bandwidth and the variation of the mode-amplitude coefficients. In particular, the time evolution of a frequency-modulated signal does not depend on mode coupling, thus possessing a behavior completely different from that of an amplitude-modulated signal.

We have set in a rigorous form a theorem describing the smoothing of mode-power fluctuations over the fiber ensemble, due to the finite bandwidth, so as to include mode-energy fluctuations in nonstationary situations.

The general case of time-dependent propagation of fluctuations has been also considered, and a closed system of equations enabling to describe it has been derived.

APPENDIX

If one defines

$$F_{mj} = \left(\frac{1}{L}\right)^{1/2}\int_{-L/2}^{L/2} dz\, K_{mj}(z)\exp[i(\beta_j - \beta_m)z] , \tag{A1}$$

one can write

$$|F_{mj}|^2 = \frac{1}{L}\int_{-L/2}^{L/2} dz'\int_{-L/2}^{L/2} dz\, \langle K_{mj}(z)K_{mj}^*(z')\rangle$$
$$\times \exp[i(\beta_j - \beta_m)(z - z')] , \tag{A2}$$

so that, remembering Eq. (31), and taking advantage of the finite range of the correlation function

$$\langle K_{mj}(z)K_{mj}^*(z')\rangle = \langle K_{mj}(z - z')K_{mj}^*(0)\rangle , \tag{A3}$$

one has

$$h_{mj} = \lim_{L\to\infty}|F_{mj}|^2 \geq 0 . \tag{A4}$$

Let us now consider Eqs. (27) and (28). Since the quantity $\langle K_{mj}(z)K_{jm}(0)\rangle$ is a real even function of z, one can write

$$\mathrm{Re}\,g_{mr} = \frac{1}{2}\sum_{j\neq m} h_{mj} + \frac{1}{2}\sum_{j\neq r} h_{jr}$$
$$+ \frac{1}{2}\lim_{L\to\infty}\left\langle\left|\left(\frac{1}{L}\right)^{1/2}\int_{-L/2}^{L/2} dz[K_{mm}(z) - K_{rr}(z)]\right|^2\right\rangle \geq 0 . \tag{A5}$$

[1] D. Marcuse, *Theory of Dielectric Optical Waveguides* (Academic, New York, 1974).
[2] S. D. Personick, "Two Derivations of the Time-Dependent Coupled-Power Equations," Bell Syst. Tech. J. 54, 47–52 (1975).
[3] R. Steinberg, "Pulse Propagation in Multimode Fibers with Frequency-Dependent Coupling," IEEE Trans. Microwave Theory Tech. MTT-23, 121–122 (1975).
[4] B. Crosignani, B. Daino, and P. Di Porto, "Statistical Coupled Equations in Lossless Optical Fibers," IEEE Trans. Microwave Theory Tech. MTT-23, 416–420 (1975).
[5] See, for example, C. H. Papas, *Theory of Electromagnetic Wave Propagation* (McGraw-Hill, New York, 1965).

[6]See, for example, A. Blanc-Lapierre and R. Fortet, *Théorie des Fonctions Aléatoires* (Masson et Cie; Paris, 1953).

[7]See, for example, M. Sargent III, M. O. Scully, and W. E. Lamb, Jr., *Laser Physics* (Addison-Wesley, Reading, Mass., 1974).

[8]B. Crosignani, B. Daino, and P. Di Porto, "Interference of Mode Patterns in Optical Fibers," Opt. Commun. 11, 178–179 (1974).

[9]B. Crosignani, B. Daino, and P. Di Porto, "Smoothing of the Mode-Power Fluctuations in Lossless Optical Fibers Due to the Source Spectral Bandwidth," Opt. Commun. 18, 551–552 (1976).

Modal dispersion in lightguides in the presence of strong coupling

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The effect of strong mode coupling on modal dispersion in optical fibers has been investigated. The pulse dispersion turns out to be qualitatively different from the one relative to the weak-coupling case, while it exhibits a drastic reduction as compared with that of the uncoupled case. The role of the initial pulse length and of the source coherence time has been elucidated.

I. INTRODUCTION

The role of mode coupling in the propagation of guided modes in a multimode lightguide is well established. It can be negative or positive, according whether the purpose is to let the lower-order modes propagate without progressively sharing their energy with the higher-order ones, or to reduce the modal dispersion of an optical fiber. While the first statement can be easily understood, the second one is not so obvious and has to be proved analytically. In fact, the existing theory[1,2] is based on a statistical approach and it describes the propagation of the *average* power in the mth mode $\langle P_m(z,t) \rangle$, where the averaging operation $\langle \ldots \rangle$ is meant to be performed over an ensemble of many macroscopically equivalent lightguides differing among themselves because of microscopically random imperfections.

In this frame it can be shown that, under suitable assumptions on the coupling, the centers of mass of the wave packets describing the $\langle P_m(z,t) \rangle$'s tend to travel with a common average velocity, while the pulse widths turn out to be proportional to $(a + bz)^{1/2}$, a and b being two positive constants, and z the traveled distance. This fact is interpreted as a mechanism of reduction of the modal dispersion in optical fibers.[1,2]

since one would expect, should the modes travel independently as they do in the absence of mode coupling, a distortion proportional to the traveled distance z.

The main limitation of the statistical approach is connected with the presence of fluctuations around the average value, which, if too ample, do not allow the average result to be confidently applied to the single fiber—with which one most often deals in practical situations. It has been demonstrated[3] that in the case in which the fiber is excited by a polychromatic source there is a fiber length $L \propto 1/T_c$ (T_c being the coherence time of the source) such that the fluctuations of the energy per mode $I_m(z,t)$:

$$I_m(z,t) = \int_{-\infty}^{\infty} P_m(z,t) \, dt \tag{1}$$

tend to vanish for z larger than L. However, this result does not concern the behavior of $P_m(z,t)$ itself, but for the stationary case in which $P_m(z,t)$ is time independent. Besides, the statistical approach does not give a correct result for the single fiber when one has to evaluate the cross-correlation term between the field $e_m(r,z,t)$ and $e_n(r,z,t)$ pertaining to the mth and nth mode, that is

$$T_{nm} = \langle e_m(r,z,t) \cdot e_n^*(r,z,t) \rangle_{av}, \qquad (2)$$

where the symbol $\langle \ldots \rangle_{av}$ indicates the averaging operation over the fluctuations of the source exciting the lightguide. As a matter of fact, it has been shown[3] that $\langle T_{nm} \rangle$ goes to zero over a distance not larger than $T_c |1/V_n - 1/V_m|^{-1}$ (V_n and V_m being the nth and mth mode group-velocity), which is just the distance over which it would go to zero in the absence of mode coupling.[4] This obviously cannot be the case for the single fiber, since the presence of a coupling certainly gives rise to a correlation between the modes over a longer distance.

The previous considerations make clear that it would be desirable to be able to find, at least for some simple workable models, an analytic solution for the problem of propagation in a *deterministic* case (that is without resorting to the statistical approach), in order to compare it with the above mentioned results. This would also allow us to evaluate in a correct way the T_{nm}'s, whose behavior may furnish a simple way of gaining information on dispersion—and thus on mode coupling.[5] In this paper, this program is carried out introducing the hypothesis of strong coupling, a case interesting *per se* which cannot be investigated by means of the statistical treatment which covers weak coupling, and considering a simple model in which only two modes interact.

In the framework of our approach, strong coupling means that the magnitude of the coupling constant K_{12} is much larger than the difference $|\beta_1 - \beta_2|$ between the propagation constants of the two unperturbed modes [see Eq. (24)]. Under this assumption, we consider the situation of resonant coupling, for which the characteristic spatial periodicity l of K_{12} if of the order of $|\beta_1 - \beta_2|^{-1}$, and that of slowly varying coupling, for which $l \gg |\beta_1 - \beta_2|^{-1}$. In both cases, propagation is significantly affected by the fact that $|K_{12}|$ is large (while, as is well known, the fulfilment of the resonant condition is essential in the weak-coupling regime[2]). Dispersion turns out to be drastically reduced with respect to the case in which coupling is absent, and its qualitative behavior differs from what one would expect according to the statitistical method, in that the pulse spreading turns out to be proportional to the fiber length z instead that to $z^{1/2}$.

The analysis of the propagation in the presence of strong coupling, besides allowing us to complete the description of the effects of mode coupling on modal dispersion, can be relevant for the study of propagation in mode scramblers, which are strongly-coupling fiber samples able to excite all guided modes in a repeatable manner.[6]

II. DESCRIPTION OF PROPAGATION IN LIGHTGUIDES

The transverse electromagnetic field propagating in a cylindrical guiding structure can be expressed as the superposition of the fields pertaining to each guided mode in the form[3,7]

$$E(r,z,t) = \sum_m e_m(r,z,t),$$

with

$$e_m(r,z,t) = E_m(r) a_m(z,t), \qquad (4)$$

where the $E_m(r)$'s represent the modes of the ideal guiding

structure (that is without mode coupling) and the expansion coefficients $a_m(z,t)$ are defined through the relation

$$a_m(z,t) = \int_{-\infty}^{\infty} c_m(z,\omega) e^{i\omega t - i\beta_m(\omega)z} d\omega. \qquad (5)$$

In Eq. (5), $\beta_m(\omega)$ is the propagation constant of the mth mode and the c_m's depend on z because of the presence of mode coupling [otherwise one would have $c_m(z,\omega) = c_m(0,\omega)$].

It is possible to show[6] that as long as only forward traveling modes are considered the transverse part of the magnetic field obeys equations identical to Eqs. (3) and (4), provided the substitution $E_m \rightarrow H_m$ is made. From the above considerations it follows that, in order to evaluate second-order averages of the kind

$$G(r_1,r_2,t_1,t_2,z) = \langle E(r_1,z,t_1) \cdot E^*(r_2,z,t_2) \rangle_{av}, \qquad (6)$$

or

$$P^\sigma(z,t) = \frac{1}{2} \mathrm{Re} \int_\sigma dr \, \langle E(r,z,t) \times H^*(r,z,t) \rangle_{av} \cdot e_z \qquad (7)$$

(e_z being a unit vector in the z direction), representing, respectively, the mutual coherence function and the power carried by the electromagnetic field through a given area σ of a fiber section, it is sufficient to investigate the behavior of the quantities

$$\langle a_m(z,t_1) a_n^*(z,t_2) \rangle_{av} \qquad (8)$$

or, equivalently, of

$$\langle c_m(z,\omega) c_n^*(z,\omega') \rangle_{av}. \qquad (9)$$

According to Eqs. (3), (4), and (7) and the mode orthogonality, it turns out[3] that $\langle a_m(z,t) a_m^*(z,t) \rangle_{av}$ is proportional to the power $P_m(z,t)$ carried by the mth mode through the whole fiber section, and that

$$P^\infty(z,t) = \sum_m P_m(z,t), \qquad (10)$$

while the nondiagonal terms $\langle a_m(z,t) a_n^*(z,t) \rangle_{av}$ can be connected with the degree of correlation between the various propagating modes. In particular one has

$$P^\sigma(z,t) = \frac{1}{2} \mathrm{Re} \sum_m \sum_n F_{mn}^\sigma \langle a_m(z,t) a_n^*(z,t) \rangle_{av}$$

$$= \mathrm{Re} \sum_m (F_{mm}^\sigma / F_{mm}^\infty) P_m(z,t)$$

$$+ \frac{1}{2} \mathrm{Re} \sum_{m \neq n} F_{mn}^\sigma \langle a_m(z,t) a_n^*(z,t) \rangle_{av}, \qquad (11)$$

with

$$F_{mn}^\sigma = \int_\sigma dr \, E_m(r) \times H_n^*(r) \cdot e_z, \qquad (12)$$

showing that the nondiagonal terms represent interference contributions between the various modes, which are always present whenever σ is finite (otherwise they disappear due to the orthogonality between different modes). Thus, the analysis of $P^\sigma(z,t)$, as compared with $P^\infty(z,t)$, can furnish a way of estimating the degree of correlation between the propagating modes.

In the absence of mode coupling, one has $c_m(z,\omega) = c_m(0,\omega)$, which, together with the approximate relation

$$\beta_m(\omega) = \beta_m(\omega_0) + (\omega - \omega_0)/V_m \qquad (13)$$

(which implies, as a necessary condition, the ratio between the bandwidth $\delta\omega$ of the propagating signal and the central frequency ω_0 to be much less than one), furnishes [see Eq. (5)]

$$a_m(z,t) = a_m(0, t - z/V_m), \qquad (14)$$

where $V_m = (d\beta_m/d\omega)_{\omega=\omega_0}^{-1}$ is the group velocity of the mth mode. This amounts to saying that $P_m(z,t)$ propagates with this velocity and that the nondiagonal terms

$$\langle a_m(z,t)a_n^*(z,t)\rangle_{av}$$
$$= \langle a_m(0, t - z/V_m)a_n^*(0, t - z/V_n)\rangle_{av} \quad (15)$$

vanish whenever the fiber length is such that they acquire a time delay larger than the coherence time $T_c = 2\pi/\delta\omega$ of the source; that is,

$$|z/V_m - z/V_n| \geq T_c. \qquad (16)$$

III. PROPAGATION IN THE PRESENCE OF MODE COUPLING: NONRESONANT CASE

In order to have an analytically solvable model, we consider a simple case in which only two modes 1 and 2 are interacting. The evolution with z of the mode amplitudes $c_1(z,\omega)$ and $c_2(z,\omega)$ is described by the following set of equations[8]:

$$\frac{dc_1(z,\omega)}{dz} = K_{12}(z)e^{i\Delta(\omega)z}c_2(z,\omega),$$
$$\qquad (17)$$
$$\frac{dc_2(z,\omega)}{dz} = -K_{12}^*(z)e^{-i\Delta(\omega)z}c_1(z,\omega),$$

where $\Delta(\omega) = \beta_1(\omega) - \beta_2(\omega)$, $K_{12}(z)$ is a z-dependent coupling coefficient and the self-coupling coefficients $K_{11}(z)$, $K_{22}(z)$, which would not give rise to any relevant effect, have been omitted for the sake of simplicity. By means of the change of dependent variables

$$c_1(z,\omega) = b_1(z,\omega)e^{i\Delta(\omega)z/2},$$
$$\qquad (18)$$
$$c_2(z,\omega) = b_2(z,\omega)e^{-i\Delta(\omega)z/2},$$

one gets from Eq. (17),

$$\frac{db_1}{dz} + i(\Delta/2)b_1 = K_{12}b_2$$
$$\qquad (19)$$
$$\frac{db_2}{dz} - i(\Delta/2)b_2 = -K_{12}^*b_1,$$

from which, by eliminating b_2, one arrives at

$$b_1'' - (K_{12}'/K_{12})b_1' + [|K_{12}|^2 + \Delta^2/4 - i(K_{12}'/2K_{12})\Delta]\,b_1 = 0, \quad (20)$$

where the prime indicates differentiation with respect to z. By performing the substitution

$$b_1 = \delta_1 \exp\left(\frac{1}{2}\int^z (K_{12}'/K_{12})\,dz\right), \qquad (21)$$

one finally obtains

$$\delta_1'' + [|K_{12}|^2 + \Delta^2/4 - i(K_{12}'/2K_{12})\Delta + (K_{12}'/2K_{12})' - (K_{12}'/2K_{12})^2]\delta_1 = 0. \quad (22)$$

Let us now assume $K_{12}(z)$ to be a slowly varying function of z. This means that there is no Fourier component of the coupling process that would provide coupling in the sense of first-order perturbation theory. However, this fact does not

preclude an effective coupling, with a consequent influence on dispersion, for $|K_{12}|$ large enough, as the following derivation will make clear. The last three terms inside the square bracket in Eq. (22) can be neglected, as compared with $|K_{12}|^2 + \Delta^2/4$, provided that the condition $l \gg |1/\Delta|$ is fulfilled, l being a characteristic oscillation length of $K_{12}(z)$. The resulting equation can be easily solved in the WKB approximation, under the usual adiabatic hypothesis

$$\frac{\dfrac{d}{dz}[(|K_{12}|^2 + \Delta^2/4)^{1/2}]}{|K_{12}|^2 + \Delta^2/4} \ll 1. \qquad (23)$$

In particular, in the strong-coupling regime,

$$|K_{12}|^2 \gg \Delta^2 \qquad (24)$$

and by assuming, for the sake of simplicity and without loss of validity of the main conclusions of our derivation, K_{12} to be independent from Z, one obtains

$$a_1(z,t) = \int d\omega \exp\{-i[\beta_1(\omega) + \beta_2(\omega)]\,z/2 + i\omega t\}$$
$$\times\, [A\exp(i|K_{12}|z + i\Delta^2z/8|K_{12}|)$$
$$+ B\exp(-i|K_{12}|z - i\Delta^2z/8|K_{12}|)], \quad (25)$$

$$a_2(z,t) = i\exp(-i\phi)\int d\omega \exp\{-i[\beta_1(\omega) + \beta_2(\omega)]z/2 + i\omega t\}$$
$$\times\, [A\exp(i|K_{12}|z + i\Delta^2z/8|K_{12}|)$$
$$- B\exp(-i|K_{12}|z - i\Delta^2z/8|K_{12}|)], \quad (26)$$

with

$$A = (\tfrac{1}{2})[c_1(0,\omega) - ie^{i\phi}c_2(0,\omega)], \qquad (27)$$
$$B = (\tfrac{1}{2})[c_1(0,\omega) + ie^{i\phi}c_2(0,\omega)],$$

and $\exp(i\phi) = K_{12}/|K_{12}|$. Assuming $c_1(0,\omega)$ and $c_2(0,\omega)$ to possess the same ω dependence, the problem is reduced to the investigation of the quantities

$$J_a(z,t) = \int d\omega\, c_1(0,\omega) \exp\{-i[\beta_1(\omega) + \beta_2(\omega)]z/2\}$$
$$\times \exp[i\omega t + i|K_{12}|z + i\Delta^2(\omega)z/8|K_{12}|], \quad (28)$$

and

$$J_b(z,t) = \int d\omega\, c_1(0,\omega) \exp\{-i[\beta_1(\omega) + \beta_2(\omega)]z/2\}$$
$$\times \exp[i\omega t - i|K_{12}|z - i\Delta^2(\omega)z/8|K_{12}|], \quad (29)$$

which is obtained from J_1 by means of the substitution $|K_{12}| \rightarrow -|K_{12}|$.

By taking advantage of Eq. (13), one can write

$$\Delta(\omega) = \Delta(\omega_0) + (\omega - \omega_0)/V^{(-)}, \qquad (30)$$

where

$$1/V^{(-)} = 1/V_1 - 1/V_2, \qquad (31)$$

which, once introduced in Eq. (29), yields

$$J_b(z,t) = e^{i\psi}\int d\omega\, c_1(0,\omega)e^{-i\omega^2\tau^2}e^{i\omega(t - z/\tilde{v}_b)} \qquad (32)$$

with

$$\psi = -[\beta_1(\omega_0) + \beta_2(\omega_0)]z/2 + \omega_0 z/2V^{(+)} - |K_{12}|z$$
$$- \Delta^2(\omega_0)z/8|K_{12}| + \Delta(\omega_0)\omega_0 z/4V^{(-)}|K_{12}|$$
$$- \omega_0^2 z/8V^{(-)2}|K_{12}| \quad (33)$$

and

$$\tau^2 = z/8V^{(-)2}|K_{12}|, \qquad (34)$$

having introduced the new velocities $V^{(+)}$ and $\bar{v}_b$ defined by the relations

$$1/V^{(+)} = 1/V_1 + 1/V_2, \tag{35}$$

$$1/\bar{v}_b = 1/2V^{(+)} + \Delta(\omega_0)/4V^{(-)}|K_{12}| - \omega_0/4V^{(-)2}|K_{12}|. \tag{36}$$

An analogous expression holds for $J_a(z,t)$ once the substitution $|K_{12}| \rightarrow -|K_{12}|$ is performed, which in particular defines another velocity

$$1/\bar{v}_a = 1/2V^{(+)} - \Delta(\omega_0)/4V^{(-)}|K_{12}| + \omega_0/4V^{(-)2}|K_{12}|. \tag{37}$$

We have now to consider the quantities $J_a J_a^*$, $J_b J_b^*$, and $J_a J_b^*$ and to average them over the fluctuations of the source (which, for a stationary source, is equivalent to taking the time average over an interval of the order of the coherence time T_c). In fact, their linear combinations furnish the significant quantities $\langle |a_1(z,t)|^2 \rangle_{av}$, $\langle |a_2(z,t)|^2 \rangle_{av}$ and $\langle a_1(z,t) a_2^*(z,t) \rangle_{av}$. Thus, according to Eq. (32), one has to evaluate the quantity

$$\langle c_1(0,\omega) c_1^*(0,\omega') \rangle_{av}. \tag{38}$$

This can be done by assuming the electric field at $z = 0$ to possess a temporal behavior of the kind

$$e^{i\omega_0 t} F(t) S(t), \tag{39}$$

where $F(t)$ is a rapidly varying function accounting for the source bandwidth while $S(t)$ represents the (usually) much slower amplitude modulation of the carrier. By assuming

$$S(t) \propto \exp(-t^2/T_p^2) \tag{40}$$

and

$$\langle F(t') F^*(t'') \rangle_{av} \propto \exp[-(t' - t'')^2/T_c^2], \tag{41}$$

where the subscripts p and c stand, respectively, for pulse and coherence, one has (see Appendix)

$$\langle c_1(0,\omega) c_1^*(0,\omega') \rangle_{av}$$
$$\propto \exp(-\Omega^2 T_1^2/4 - \Omega'^2 T_1^2/4 + \Omega\Omega' T_2^2/2), \tag{42}$$

where

$$T_1^2 = T_p^2(T_p^2 + T_c^2)/(2T_p^2 + T_c^2) \tag{43}$$

and

$$T_2^2 = T_p^4/(2T_p^2 + T_c^2), \tag{44}$$

with $\Omega = \omega - \omega_0$ and $\Omega' = \omega' - \omega_0$. In particular, the monochromatic and the stationary cases are respectively recovered by letting T_c and T_p become infinitely large.

By taking advantage of Eq. (42), one can write

$$\langle J_b(z,t) J_b^*(z,t) \rangle_{av} \propto \int d\omega \int d\omega' \exp[-\omega^2(i\tau^2 + T_1^2/4) - \omega'^2(-i\tau^2 + T_1^2/4)]$$
$$\times \exp\{\omega[it_b + \omega_0(T_1^2 - T_2^2)/2] + \omega'[-it_b + \omega_0(T_1^2 - T_2^2)/2 + \omega T_2^2/2]\} \exp[-\omega_0^2(T_1^2 - T_2^2)/2], \tag{45}$$

with $t_b = t - z/\bar{v}_b$. After performing the two integrals over ω and ω', Eq. (45) furnishes

$$\langle J_b J_b^* \rangle_{av} \propto (1/T) \exp[-(t - z/v_b)^2/T^2] \tag{46}$$

where

$$T = (T_0^2/2 + 8\tau^4/T_3^2)^{1/2}, \tag{47}$$

$$1/v_b = 1/2V^{(+)} + \Delta(\omega_0)/4V^{(-)}|K_{12}|, \tag{48}$$

with

$$T_3^2 = T_1^2 - T_2^2 = T_p^2 T_c^2/(2T_p^2 + T_c^2), \tag{49}$$

$$T_0^2 = T_1^2 + T_2^2 = (2T_p^4 + T_p^2 T_c^2)/(2T_p^2 + T_c^2). \tag{50}$$

An analogous expression holds true for $\langle J_a J_a^* \rangle_{av}$, which reads

$$\langle J_a J_a^* \rangle_{av} \propto (1/T) \exp[-(t - z/v_a)^2/T^2], \tag{51}$$

where

$$1/v_a = 1/2V^{(+)} - \Delta(\omega_0)/4V^{(-)}|K_{12}|. \tag{52}$$

One can observe that the group-velocities v_a and v_b could have been also directly obtained by taking, respectively, the derivative with respect to ω (evaluated at $\omega = \omega_0$) of the propagation constants of the first and second terms contributing to a_1 [Eq. (25)] or a_2 [Eq. (26)].

The term $\langle J_a J_b^* \rangle_{av}$ cannot be expressed in the form of a wave packet propagating with a definite velocity, and will be discussed later on.

The expression of $\langle J_a J_a^* \rangle_{av}$ and $\langle J_b J_b^* \rangle_{av}$, as given by Eqs. (46) and (51), shows that the signal inside the fiber evolves in two distinct pulses, traveling with two slightly different velocities v_a and v_b. Besides, the temporal width of each pulse increases with z, so that one has to take into account two different sources of dispersion. In practice, this second kind of dispersion can be neglected for the values of z such that [see Eq. (47)]

$$T^2(z)/T_3 < T_0/4 \simeq T_p/4. \tag{53}$$

The other contribution can be put in a quantitative form by introducing the time delay T_d between the centers of mass of the two pulses at a distance z; that is,

$$T_d(z) = z(1/v_b - 1/v_a) = \Delta(\omega_0)z/2V^{(-)}|K_{12}|. \tag{54}$$

It is immediate to see that this kind of dispersion is dominant—the distance between the centers of mass of the two pulses increasing more rapidly than their widths. In fact, it follows from Eqs. (34) and (54) that

$$T_d T_3/\tau^2 = 4V^{(-)}\Delta(\omega_0)T_3$$
$$= 4\omega_0 T_3 V^{(-)}\Delta(\omega_0)/\omega_0 \simeq 4\omega_0 T_3, \tag{55}$$

having taken into account the circumstance that the difference between the inverse of group and phase velocities, $V^{(-)}$ and $\omega_0/\Delta(\omega_0)$, are of the same order.[9] Observing that T_3 is of the order of the coherence time, the ratio between T_d and τ^2/T_3 turns out to be $\omega_0/\delta\omega \gg 1$.

Returning now to the problem of evaluating the influence of the term $\langle J_a J_b^* \rangle_{av}$, it may be noted that its importance tends to become negligible for distances such that

$$T_d(z) > T_p, \tag{56}$$

after which the two pulses do not overlap anymore.

IV. PROPAGATION IN THE PRESENCE OF MODE COUPLING: RESONANT CASE

The resonant case corresponds to a sinusoidally modulated coupling constant

$$K_{12}(z) = 2K \cos(\chi z), \tag{57}$$

with

$$\chi = \Delta(\omega_0), \tag{58}$$

the (resonance) frequency ω_0 having been assumed to coincide with the central frequency of the exciting source. By introducing Eq. (57) into Eqs. (17) one obtains

$$\frac{dc_1(z,\omega)}{dz} = K e^{-i\chi z + i\Delta(\omega)z} c_2(z,\omega),$$

$$\frac{dc_2(z,\omega)}{dz} = -K^* e^{i\chi z - i\Delta(\omega)z} c_1(z,\omega), \tag{59}$$

having neglected in the right-hand side of Eqs. (59) the terms containing the rapidly oscillating factors $\exp\{\pm i[\Delta(\omega_0) + \chi]z\}$ with respect to the slowly varying terms containing $\exp\{\pm i[\Delta(\omega_0) - \chi]z\}$, whenever $|K| \ll |\chi + \Delta(\omega)|$. By using this approximation, it is possible to write

$$K_{12} = K e^{-i\Delta(\omega_0)z}. \tag{60}$$

One can now use the procedure of Sec. III, thus obtaining

$$a_1(z,t) = \exp\left(\frac{-i\Delta(\omega_0)z}{2}\right) \int d\omega$$

$$\times \exp\left(-i\frac{\beta_1(\omega) + \beta_2(\omega)}{2}z + i\omega t\right)$$

$$\times \left[A \exp\left(i|K|z + \frac{i\Delta^2(\omega)z}{8|K|} - \frac{i\Delta(\omega_0)\Delta(\omega)z}{4|K|}\right.\right.$$

$$\left. + \frac{i\Delta^2(\omega_0)z}{8|K|}\right) + B \exp\left(-i|K|z - \frac{i\Delta^2(\omega)z}{8|K|}\right.$$

$$\left.\left. + \frac{i\Delta(\omega_0)\Delta(\omega)z}{4|K|} - \frac{i\Delta^2(\omega_0)z}{8|K|}\right)\right] \tag{61}$$

$$a_2(z,t) = i \exp\left(-i\Phi + \frac{i\Delta(\omega_0)z}{2}\right) \int d\omega$$

$$\times \exp\left(-i\frac{\beta_1(\omega) + \beta_2(\omega)}{2}z + i\omega t\right)$$

$$\times \left[A \exp\left(i|K|z + \frac{i\Delta^2(\omega)z}{8|K|} - \frac{i\Delta(\omega_0)\Delta(\omega)z}{4|K|} + \frac{i\Delta^2(\omega_0)z}{8|K|}\right)\right.$$

$$\left. - B \exp\left(-i|K|z - \frac{i\Delta^2(w)z}{8|K|} + \frac{i\Delta(\omega_0)\Delta(\omega)z}{4|K|} - \frac{i\Delta^2(\omega_0)z}{8|K|}\right)\right], \tag{62}$$

with

$$A = (\tfrac{1}{2})[c_1(0,\omega) - ie^{i\Phi}c_2(0,\omega)], \tag{63}$$

$$B = (\tfrac{1}{2})[c_1(0,\omega) + ie^{i\Phi}c_2(0,\omega)],$$

where $\exp(i\Phi) = K/|K|$.

The expressions of $a_1(z,t)$ and $a_2(z,t)$ show that the main properties of propagation can be deduced following the derivation of Sec. III. In the present resonant case, the group-velocities v_a and v_b of both contributions to a_1 (and a_2) coincide, since

$$\frac{1}{v_{a,b}} = \frac{d}{d\omega}\left(\frac{\beta_1(\omega) + \beta_2(\omega)}{2} \mp \frac{\Delta^2(\omega)}{8|K|} \pm \frac{\Delta(\omega_0)\Delta(\omega)}{4|K|}\right)_{\omega=\omega_0}$$

$$= \frac{1}{2}\left(\frac{1}{V_1} + \frac{1}{V_2}\right), \tag{64}$$

so that the signal appears in the form of a single pulse undergoing a temporal broadening T given by Eq. (47), which for large z reduces to

$$T = 8^{1/2}\frac{\tau^2}{T_3} = \frac{1}{8^{1/2}}\frac{z}{V^{(-)2}|K_{12}|T_3}. \tag{65}$$

In other words, the two distinct pulses of the nonresonant situation overlap, so that the only cause of dispersion is given by their common broadening.

V. INFLUENCE OF MODE COUPLING ON DISPERSION

One can compare the results of the previous sections with the one obtained in the frame of the statistical-coupled power theory, according to which the temporal width of the pulse T_s is proportional to the square root of z, that is[10]

$$T_s = (2z)^{1/2}/V^{(-)}h^{1/2}, \tag{66}$$

where

$$h = \int_{-\infty}^{\infty} d\zeta \langle K_{12}(\zeta)K_{12}^*(0)\rangle e^{i\Delta(\omega_0)\zeta}, \tag{67}$$

the angular brackets indicating the averaging operation over an ensemble of many macroscopically similar fibers. Equations (54), (65), and (66) show that the results of the weak- and strong-coupling cases are qualitatively different. In particular, the deterministic approach brings under consideration the role of the coherence time T_c of the exciting source and of the pulse duration T_p. It provides a dispersion which exhibits the same linear z dependence of the uncoupled case, for which the width T_u of the signal carried by the two uncoupled modes is given by[10]

$$T_u = z/V^{(-)}, \tag{68}$$

but which is quantitatively very different. In fact, one has,

$$T_d/T_u = \Delta(\omega_0)/2|K_{12}| \ll 1 \tag{69}$$

and

$$T/T_u \ll T_d/T_u = \Delta(\omega_0)/2|K_{12}| \ll 1, \tag{70}$$

where the first inequality in Eq. (70) follows from Eq. (55), in agreement with the intuitive statement that the presence of a strong coupling must considerably reduce dispersion, in both resonant and nonresonant situations.

We wish to stress again the fact that the present deterministic method deals with a case in which the c_m's considerably vary, in the nonresonant case, over a distance of the order $1/\Delta(\omega_0)$ (strong coupling), while the opposite hypothesis is introduced in order to deal with the statistical approach.[3]

Actually, the strong-coupling hypothesis has as a consequence that the powers per mode $P_1(z,t)$ and $P_2(z,t)$ possess the same temporal evolution. Accordingly, two initially correlated modes travel without acquiring any mutual time delay, so that $\langle a_1(z,t)a_2^*(z,t)\rangle_{av}$ does not vanish after the distance defined by Eq. (16), which implies that the transverse spatial correlation between them is preserved over a very long traveled path.

VI. CONCLUSIONS

We have treated the propagation of a pulse of given initial duration, injected into the fiber by a source with definite temporal coherence properties, by adopting a simple model in which only two modes are considered. This has been performed under the assumption of strong coupling, by means of a deterministic approach, which is able to cover both the resonant and far-from-resonance cases.

Strong coupling is effective in both resonant and nonresonant conditions and affects qualitatively and quantitatively the behavior of dispersion. Far from resonance, dispersion is associated with the breaking up of the initial signal into two distinct pulses proceeding with different velocities, such that the corresponding mutual delay exceeds the temporal broadening of the single pulse. Conversely, this broadening turns out to be the only source of dispersion in the resonant case, where the signal travels as a single pulse.

In both cases, dispersion is drastically reduced with respect to the uncoupled situation, and the main qualitative difference with respect to the weak-coupling regime consists in the linear dependence of pulse dispersion on the traveled length.

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APPENDIX

According to Eqs. (5) and (39), one has

$$c_1(0,\omega) \propto \int_{-\infty}^{\infty} dt'\, e^{-i\omega t'} S(t')F(t')\, e^{i\omega_0 t'}, \quad (A1)$$

so that

$$\langle c_1(0,\omega)c_1^*(0,\omega')\rangle_{av} \propto \int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} dt''\, S(t')S^*(t'')$$
$$\times e^{-i\Omega t' + i\Omega' t''}\langle F(t')F^*t'')\rangle_{av}, \quad (A2)$$

where $\Omega = \omega - \omega_0$ and $\Omega' = \omega' - \omega_0$. After introducing the time Fourier-transform $\hat{S}(f)$ of the slowly-varying amplitude $S(t)$, one can take advantage of the stationariety of the rapidly-varying part $F(t)$

$$\langle F(t')F^*(t'')\rangle_{av} = G(t' - t''), \quad (A3)$$

thus being able to rewrite Eq. (A2) in the form

$$\langle c_1(0,\omega)c_1^*(0,\omega')\rangle_{av} \propto \int_{-\infty}^{\infty} df$$
$$\times \int_{-\infty}^{\infty} d\tau\, \hat{S}(f)\hat{S}(f + \Omega' - \Omega)G(\tau)e^{-i(\Omega-f)\tau}. \quad (A4)$$

By recalling that, according to Eqs. (40) and (41), one has

$$\hat{S}(f) \propto \exp(-f^2 T_p^2/4) \quad (A5)$$

and

$$\int_{-\infty}^{\infty} d\tau \exp[-i(\Omega - f)\tau]\, G(\tau) \propto \exp[-(\Omega - f)^2 T_c^2/4],$$
$$(A6)$$

it is possible to perform the integration in Eq. (A4), thus obtaining Eq. (42).

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[1]S. D. Personick, "Time Dispersion in Dielectric Waveguides," Bell Syst. Tech. J. 50, 843–859 (1971).
[2]see, also, D. Marcuse, *Theory of Dielectric Optical Waveguides,* (Academic, New York, 1974).
[3]B. Crosignani, P. Di Porto, and C. H. Papas, "Theory of time-dependent propagation in multimode lightguides," J. Opt. Soc. Am. 67, 1300–1307 (1977).
[4]B. Crosignani and P. Di Porto, "Coherence of an electromagnetic field propagating in a weakly guiding fiber," J. Appl. Phys. 44, 4616–4617 (1973).
[5]B. Crosignani and P. Di Porto, "Propagation of Coherence and Very High Resolution Measurements in Optical Fibers," in Fibers & Integrated Optics, S.P.I.E., Vol. 17, (1976).
[6]see, for example, M. Ikeda, Y. Murakami, and K. Kitayami, "Mode scrambler for optical fibers," Appl. Opt. 16, 1045–1049 (1977).
[7]See Ref. 2, p. 101.
[8]See Ref. 2, p. 105.
[9]See Ref. 2, p. 79.
[10]See Ref. 2, p. 212.

Temporal spreading of a pulse propagating in a two-mode optical fiber

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The dependence of the temporal width of the impulse response on the length z of a two-mode optical fiber is examined. This quantity, which is proportional to z in the absence of mode coupling and to $z^{1/2}$ in the presence of weak random coupling among the guided modes, possesses a different dependence in the case of a deterministic resonant-coupling model, appropriate for describing a rather general class of actual situations. The relevant role played by the coherence time of the signal is demonstrated.

INTRODUCTION

The influence on modal dispersion of coupling among the guided modes of a multimode optical fiber has been examined by many authors[1] since a possible reduction of the pulse spreading due to this mechanism was first predicted.[2] The analysis of electromagnetic propagation in a fiber in the presence of mode coupling is usually accomplished in the frame of a statistical theory which deals with quantities such as the powers P_n of the individual modes averaged over a suitable ensemble of similar fibers.[3] This approach allows one to investigate the electromagnetic propagation by means of a set of differential coupled equations describing the evolution of the averaged quantities $\langle P_n \rangle$, which, under certain hypotheses, can be cast in an exactly solvable form.[4] One of the main results of this theory concerns modal dispersion. More precisely, it turns out that the width of the impulse response exhibits, for a large traveled distance z, a square-root-type dependence on z, while the width depends linearly on z in the uncoupled case.[5] This result implies that for long traveled distances mode coupling reduces dispersion. Reduction of dispersion due to mode coupling has actually been observed.[6]

The statistical approach, which has some intrinsic limitations due to its very nature,[7] is suitable for dealing with weak-coupling situations such as those arising from the presence of many small unpredictable imperfections in the fiber structure.[8] However, there are fibers in which mode coupling takes place in such a way that they cannot be considered as typical samples of the ensemble to which the statistical theory applies. This occurs, in particular, in connection with spatially quasiperiodic imperfections giving rise to a resonant coupling of the various modes. In this paper, we wish to examine electromagnetic propagation in a two-mode optical fiber possessing this kind of coupling, our main aim being the investigation of the dependence of pulse spreading on length traveled. This spreading turns out to exhibit a behavior that is different from the square-root-type dependence predicted by statistical theory.

The two-mode optical fiber, beyond furnishing the simplest model for treating mode coupling, is interesting *per se* due to the recent developments of single-mode optical waveguides in which two effective modes with orthogonal polarization are present.[9,10]

I. PROPAGATION IN THE PRESENCE OF RESONANT COUPLING WITH CONSTANT AMPLITUDE

The propagation of a finite-bandwidth field

$$\mathbf{E}(\mathbf{r},z,t) = \mathbf{E}_1(\mathbf{r}) \int c_1(z,\omega) e^{i\omega t - i\beta_1(\omega)z} d\omega$$

$$+ \mathbf{E}_2(\mathbf{r}) \int c_2(z,\omega) e^{i\omega t - i\beta_2(\omega)z} d\omega$$

$$= \mathbf{E}_1(\mathbf{r})a_1(z,t) + \mathbf{E}_2(\mathbf{r})a_2(z,t) \quad (1)$$

in a lossless fiber supporting two guided coupled modes is

governed by the system of equations[3,7]

$$\frac{d}{dz} c_1(z,\omega) = K_{12}(z)e^{i\Delta(\omega)z}c_2(z,\omega)$$

$$\frac{d}{dz} c_2(z,\omega) = -K_{12}^*(z)e^{-i\Delta(\omega)z}c_1(z,\omega), \tag{2}$$

which describes the evolution of the mode amplitudes $c_i(z,\omega)$. We have indicated with $\mathbf{r}$ and z the transverse and longitudinal coordinates, with $\Delta(\omega) = \beta_1(\omega) - \beta_2(\omega)$ the difference between the propagation constants of the two unperturbed modes $\mathbf{E}_1(\mathbf{r})$ and $\mathbf{E}_2(\mathbf{r})$, and with $K_{12}(z)$ the coupling coefficient, which is related to the departure of the fiber structure from the ideal configuration.

If we assume $K_{12}(z)$ to be of the form

$$K_{12}(z) = 2K\cos(\chi z), \tag{3}$$

with

$$\chi = \Delta(\omega_0), \tag{4}$$

ω_0 representing a frequency within the spectral width $\delta\omega$ of the signal, the set of Eqs. (2) becomes

$$\frac{d}{dz} c_1(z,\omega) = Ke^{i[\Delta(\omega)+\Delta(\omega_0)]z}c_2(z,\omega)$$
$$+ Ke^{i[\Delta(\omega)-\Delta(\omega_0)]z}c_2(z,\omega), \tag{5}$$

$$\frac{d}{dz} c_2(z,\omega) = -K^*e^{-i[\Delta(\omega)-\Delta(\omega_0)]z}c_1(z,\omega)$$
$$- K^*e^{-i[\Delta(\omega)+\Delta(\omega_0)]z}c_1(z,\omega).$$

By taking advantage of the rotating wave approximation which, whenever $|K| \ll |\Delta(\omega)|$, allows us to neglect the rapidly oscillating terms of the kind $\exp\{\pm i[\Delta(\omega) + \Delta(\omega_0)]z\}$ that average out to zero over the distance of propagation, the set of Eqs. (5) reduces to

$$\frac{d}{dz} c_1(z,\omega) = Ke^{i\gamma(\omega)z}c_2(z,\omega),$$

$$\frac{d}{dz} c_2(z,\omega) = -K^*e^{-i\gamma(\omega)z}c_1(z,\omega), \tag{6}$$

with $\gamma(\omega) = \Delta(\omega) - \Delta(\omega_0)$. The set of Eqs. (6) can be easily solved and the resulting expressions for $c_1(z,\omega)$ and $c_2(z,\omega)$ inserted into Eq. (1), thus giving[11]

$$a_1(z,t) = \exp(-i\Delta(\omega_0)z/2)$$

$$\times \int_{-\infty}^{+\infty} \exp\{-i[\beta_1(\omega) + \beta_2(\omega)]z/2 + i\omega t\}$$

$$\times \{A\exp[i|K|z + i(\omega - \omega_0)^2 z/8|K|\bar{v}^2]$$
$$+ B\exp[-i|K|z - i(\omega - \omega_0)^2 z/8|K|\bar{v}^2]\}d\omega \tag{7}$$

and

$$a_2(z,t) = i\exp[-i\phi + i\Delta(\omega_0)z/2]$$

$$\times \int_{-\infty}^{+\infty} \exp\{-i[\beta_1(\omega) + \beta_2(\omega)]z/2 + i\omega t\}$$

$$\times \{A\exp[i|K|z + i(\omega - \omega_0)^2 z/8|K|\bar{v}^2]$$
$$- B\exp[-i|K|z - i(\omega - \omega_0)^2 z/8|K|\bar{v}^2]\}d\omega. \tag{8}$$

Here,

$$A = (1/2)[c_1(0,\omega) - ie^{i\phi}c_2(0,\omega)],$$
$$B = (1/2)[c_1(0,\omega) + ie^{i\phi}c_2(0,\omega)], \tag{9}$$

where $\exp(i\phi) = K/|K|$ and

$$1/\bar{v} = 1/V_1 - 1/V_2, \tag{10}$$

$V_i^{-1} = (d\beta_i/d\omega)_{\omega=\omega_0}$ indicating the inverse of the group velocity of the ith unperturbed mode. Furthermore, we have assumed

$$\delta\omega \ll |K|\bar{v} \tag{11}$$

and

$$\Delta(\omega) \simeq \Delta(\omega_0) + (\omega - \omega_0)/\bar{v}, \tag{12}$$

$\delta\omega$ being the source bandwidth.

The electromagnetic powers $P_1(z,t)$ and $P_2(z,t)$ carried by modes 1 and 2, whose behavior determines the pulse shape, are proportional to $\langle|a_1(z,t)|^2\rangle_{av}$ and $\langle|a_2(z,t)|^2\rangle_{av}$, where the symbol $\langle\ \rangle_{av}$ indicates the averaging operation over the fluctuations of the source exciting the fiber. In order to determine $\langle|a_1(z,t)|^2\rangle_{av}$ and $\langle|a_2(z,t)|^2\rangle_{av}$, it is necessary to evaluate the quantities $\langle c_1(0,\omega)c_1^*(0,\omega')\rangle_{av}$, $\langle c_2(0,\omega)c_2^*(0,\omega')\rangle_{av}$, and $\langle c_1(0,\omega)c_2^*(0,\omega')\rangle_{av}$ as functions of the source bandwidth and pulse duration. By assuming a pulse of Gaussian shape and width T_p injected into the fiber by a source possessing a coherence time $T_c = 2\pi/\delta\omega$, one has[11]

$$\langle c_1(0,\omega)c_1^*(0,\omega')\rangle_{av} = \langle c_1(0,\omega_0)c_1^*(0,\omega')\rangle_{av}$$
$$\times \exp(-\Omega^2 T_1^2/4 - \Omega'^2 T_1^2/4 + \Omega\Omega' T_2^2/2) \equiv F(\Omega,\Omega'), \tag{13}$$

where

$$T_1^2 = T_p^2(T_p^2 + T_c^2)/(2T_p^2 + T_c^2) \tag{14}$$

and

$$T_2^2 = T_p^4/(2T_p^2 + T_c^2), \tag{15}$$

with $\Omega = \omega - \omega_0$ and $\Omega' = \omega' - \omega_0$.

With the help of Eqs. (7), (8), and (13) we can write, after assuming for the sake of simplicity that $c_1(0,\omega) = c_2(0,\omega)$,

$$\langle|a_1(z,t)|^2\rangle_{av} = \int_{-\infty}^{+\infty} d\Omega \int_{-\infty}^{+\infty} d\Omega' e^{i(\Omega-\Omega')t}$$
$$\times H(z,\Omega,\Omega';K)F(\Omega,\Omega'), \tag{16}$$

where

$$H(z,\Omega,\Omega';K) = (1/2)[(1 + \sin\phi)e^{i(\Omega^2-\Omega'^2)\tau^2(z)}$$
$$+ (1 - \sin\phi)e^{i(\Omega^2-\Omega'^2)\tau^2(z)}$$
$$+ i\cos\phi e^{-2i|K|z} e^{-i(\Omega^2+\Omega'^2)\tau^2(z)}$$
$$- i\cos\phi e^{2i|K|z} e^{i(\Omega^2+\Omega'^2)\tau^2(z)}] \tag{17}$$

and

$$\tau^2(z) = z/8|K|\bar{v}^2, \tag{18}$$

the same expression being valid for $\langle|a_2(z,t)|^2\rangle_{av}$ but for the substitution $\cos\phi \rightarrow -\cos\phi$.

By performing the double integral appearing in Eq. (16), one finally obtains

$$\langle|a_{1,2}(z,t)|^2\rangle_{av} = \frac{4\pi\langle c_1(0,\omega_0)c_1^*(0,\omega_0)\rangle_{av}}{T_3(T_0^2 + 16\tau^4(z)/T_3^2)^{1/2}} \exp\{-2t'^2/[T_0^2 + 16\tau^4(z)/T_3^2]\} \pm \frac{4\pi\cos\phi\langle c_1(0,\omega_0)c_1^*(0,\omega_0)\rangle_{av}}{\{[T_3^2T_0^2 - 16\tau^4(z)]^2 + 64T_1^4\tau^4(z)\}^{1/4}}$$

$$\times \exp(-2t'^2[T_3^4T_0^2 + 16T_0^2\tau^4(z)]/\{[T_3^2T_0^2 - 16\tau^4(z)]^2 + 64T_1^4\tau^4(z)\})$$

$$\times \sin(2t'^2[4T_3^4\tau^2(z) + 64\tau^6(z)]/\{[T_3^2T_0^2 - 16\tau^4(z)]^2 + 64T_1^4\tau^4(z)\} + 2|K|z + \psi/2), \quad (19)$$

where

$$t' = t - z/V, \qquad (20)$$

with

$$1/V = (1/V_1 + 1/V_2)/2, \qquad (21)$$

$$t_3^2 = T_1^2 - T_2^2 = T_p^2T_C^2/(2T_p^2 + T_C^2), \qquad (22)$$

$$T_0^2 = T_1^2 + T_2^2 = T_p^2, \qquad (23)$$

and

$$\psi = \arg[T_3^2T_0^2 - 16\tau^4(z) - 8iT_1^2\tau^2(z)]. \qquad (24)$$

The significant result contained in Eq. (19) is that the powers $P_1(z,t)$ and $P_2(z,t)$ exhibit the same space-time evolution, traveling with a bulk velocity V and undergoing a broadening which, for z sufficiently large ($z > \bar{z}$, $\bar{z}$ being defined through the relation $\tau^2(\bar{z}) = T_PT_3/4$), is proportional to the traveled length itself.

Actually, it is the first term on the right-hand side of Eq. (19) that plays the leading role for what concerns broadening, since $T_3 < T_p$ [see Eqs. (22) and (23)]. In practice, $T_c \ll T_p$, so that $T_3 \simeq T_c/\sqrt{2} \ll T_p$. Accordingly, the temporal width $T_p(z)$ of the pulse can be evaluated as

$$T_p(z) = z/(2|K|\bar{v}^2T_c)^{1/2} \quad \text{for } z > \bar{z}, \qquad (25)$$

so that the pulse temporal width behaves linearly with z, as in the absence of mode coupling.

The width $T_p(z)$ can be easily shown to be much smaller than $T = z/\bar{v}$ relative to the noncoupling case, that is, to the difference between the group delays of the two unperturbed modes. As a matter of fact, one has

$$T_p(z) = (T/\sqrt{2})(\delta\omega/2\pi|K|\bar{v}), \qquad (26)$$

so that, according to Eq. (11), $T_p(z) \ll T$. This fact confirms the positive role played by coupling in reducing dispersion. Thus, the influence of mode coupling on pulse broadening is only of a quantitative nature in our deterministic model, while, in the framework of the statistical theory, mode coupling results in a z dependence of the pulse width of the square-root type. This latter kind of dependence can be accounted for in an intuitive way by taking advantage of the Central Limit Theorem in connection with the random "jumps" of each photon from one mode to the other, while no random process occurs in our model.

We wish to conclude this section by noting the relevance of the coherence time T_c of the source [provided by Eq. (11) is verified] in determining the pulse spreading in our case, while T_c (or, equivalently, the source bandwidth) does not influence the pulse shape according to the conclusions of the statistical approach.[3] This is directly related to the basic hypothesis underlying the statistical theory, according to which the field amplitudes do not vary appreciably over the small distance after which the mode-coupling random imperfections decorrelate, leading to a system of coupled equations involving only the mode powers $\langle P_n\rangle$.[3] On the contrary, coherence effects are preserved in our situation due to the periodicity of the coupling coefficient, a fact that does not allow consideration of our fiber as a typical sample of the statistical ensemble.

II. INFLUENCE OF VARIATIONS OF COUPLING AMPLITUDE AND PERIODICITY

A more general coupling model can be obtained by letting the amplitude K and the periodicity χ of $K_{12}(z)$ to assume different constant values K_i and χ_i over different fiber regions (possessing, in general, different lengths L_i). In order to preserve the resonance condition, we assume the existence of frequencies ω_i lying inside the bandwidth $\delta\omega$ such that

$$\chi_i = \beta_1(\omega_i) - \beta_2(\omega_i). \qquad (27)$$

From an analytical point of view, the problem of pulse propagation in this structure can be easily solved by observing that, due to the form of Eq. (16), the quantity

$$H(L_1,\Omega,\Omega';K_1)F(\Omega,\Omega') \qquad (28)$$

furnishes the "boundary condition" for deriving the solution in the second region, which reads

$$\langle|a_1(z,t)|^2\rangle_{av} = \int_{-\infty}^{+\infty}d\Omega\int_{-\infty}^{+\infty}d\Omega'e^{i(\Omega-\Omega')t}$$

$$\times H(z_2,\Omega,\Omega';K_2)H(L_1,\Omega,\Omega';K_1)F(\Omega,\Omega'), \qquad (29)$$

for $0 \le z_2 \le L_2$ and $z = L_1 + z_2$. More in general, one has

$$\langle|a_1(z,t)|^2\rangle_{av} = \int_{-\infty}^{+\infty}d\Omega\int_{-\infty}^{+\infty}d\Omega'e^{i(\Omega-\Omega')t}$$

$$\times H(z_n,\Omega,\Omega';K_n)H(L_{n-1},\Omega,\Omega';K_{n-1})$$

$$\times H(L_{n-2},\Omega,\Omega';K_{n-2}) \times \cdots H(L_1,\Omega,\Omega';K_1)F(\Omega,\Omega'), \qquad (30)$$

for $0 \le z_n \le L_n$ and $z = L_1 + L_2 + \cdots + L_{n-1} + z_n$.

The right-hand side of Eq. (30) is the sum of a number of terms of the kind

$$\int_{-\infty}^{+\infty}d\Omega\int_{-\infty}^{+\infty}d\Omega'\exp(ia\Omega^2 + ib\Omega'^2)$$

$$\times \exp[i(\Omega - \Omega')(t - z/V)]F(\Omega,\Omega'), \qquad (31)$$

where a and b are linear combinations of the quantities $\tau^2(L_1)$, $\tau^2(L_2),\ldots,\tau^2(z_n)$ with coefficients $+1$ and -1 in all possible combinations. By recalling Eq. (13), the double integral appearing in Eq. (31) can be explicitly evaluated. The most relevant contributions to $\langle|a_1(z,t)|^2\rangle_{av}$ come from the terms for which

$$a = -b = \pm[\tau^2(L_1) + \tau^2(L_2) + \cdots + \tau^2(z_n)], \qquad (32)$$

which evolve in the form of a pulse moving with a velocity V and possessing a temporal width $T_p(z) = T_p(L_1 + L_2 + \cdots$

$+ L_{n-1} + z_n$) given by

$$T_p(z)$$

$$= \frac{L_1/|K_1| + L_2/|K_2| + \cdots + L_{n-1}/|K_{n-1}| + z_n/|K_n|}{\sqrt{2}\,\partial^2 T_c}.$$

(33)

Equation (33) generalizes Eq. (25) but assures a linear dependence of $T_p(z)$ on the traveled length only inside each single fiber region with a given value of $|K|$.

III. CONCLUSIONS

We have investigated the broadening of a pulse propagating in a two-mode fiber, made up of a number of regions in each of which a spatially periodic resonant coupling with constant amplitude is present. This kind of optical waveguide does not belong to the statistical ensemble considered in the random-coupling theory, so that electromagnetic propagation cannot be treated by means of coupled-power equations, and the mode-amplitude evolution has to be explicitly investigated. As a consequence, the temporal broadening of the pulse turns out to be related to the source coherence time, while it does not exhibit the dependence on the square root of the fiber length predicted by the statistical approach.

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[1]See, for example, R. Olshansky, "Mode coupling effects in graded-index optical fibers," Appl. Opt. 14, 935–945 (1975) and bibliography therein.
[2]S. D. Personick, "Time dispersion in dielectric waveguides," Bell Syst. Tech. J. 50, 843–859 (1971).
[3]D. Marcuse, Theory of Dielectric Optical Waveguides (Academic, New York, 1974).
[4]D. Gloge, "Optical power flow in multimode fibers," Bell Syst. Tech. J. 51, 1767–1783 (1972).
[5]This result holds true for a single fiber; for the situation of many fibers jointed together see, for example, M. Eve, "Multipath time dispersion theory of an optical network," Opt. Quantum Electron. 10, 41–51 (1978).
[6]E. L. Chinnock, L. G. Cohen, W. S. Holden, R. D. Standley, and D. B. Keck, "The length dependence of pulse spreading in the CGW-Bell-10 optical fiber," Proc. IEEE (Lett.) 61, 1499–1500 (1973).
[7]For a detailed discussion of this point, see B. Crosignani, P. Di Porto, and C. H. Papas, "Theory of time-dependent propagation in multimode lightguides," J. Opt. Soc. Am. 67, 1300–1307 (1977).
[8]The weak-coupling hypothesis also underlies the results of the ray theory; see, for example, J. Arnaud and M. Rousseau, "Ray theory of randomly modulated optical fibers," Opt. Lett. 3, 63–65 (1978).
[9]S. C. Rashleigh and R. Ulrich, "Polarization mode dispersion in single-mode fibers," Opt. Lett. 3, 60–62 (1978).
[10]R. H. Stolen, V. Ramaswamy, P. Kaiser, and W. Pleibel, "Linear polarization in birefringent single-mode fibers," Appl. Phys. Lett. 33, 699–701 (1978).
[11]B. Crosignani, P. Di Porto, and C. H. Papas, "Modal dispersion in lightguides in the presence of strong coupling," J. Opt. Soc. Am. 68, 1586–1591 (1978).
[12]S. Kawakami and M. Ikeda, "Transmission characteristics of a two-mode optical waveguide," IEEE J. Quantum Electron. QE-14, 608–614 (1978).

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Optical Fibres, Integrated Optics and Their Military Applications

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FINITE-BANDWIDTH PROPAGATION IN MULTIMODE OPTICAL FIBERS

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SUMMARY

The propagation in a multimode optical fiber of a finite-bandwidth optical carrier modulated by a nonstationary signal is investigated. The fluctuations of the field due to random mode-coupling are considered and the set of coupled equations describing their evolution is derived. In particular, this allows us to investigate the propagation of a frequency-modulated signal and to obtain a general theorem concerning the asymptotic behavior of mode-power fluctuations.

1. INTRODUCTION

The propagation of a stationary monochromatic carrier in a multimode optical fiber in the presence of mode-coupling has been extensively studied by Marcuse (1), in the frame of a statistical approach based on the introduction of an ensemble of similar fibers slightly differing from a common ideal structure for the presence of random imperfections. The procedure allows one to write two closed systems of differential coupled equations for the single-mode powers averaged over the ensemble of fibers $\langle P_m \rangle$ and for the correlations $\langle P_m P_n \rangle$ (1),(2). The extension to the nonstationary nonmonochromatic case has been accomplished, for what concerns the average powers $\langle P_m \rangle$, on an intuitive basis by Marcuse (1) and by means of more rigorous treatments by Personick (3) and Steinberg (4).

We generalize the statistical approach to furnish the evolution of the quantities $\langle \overline{c_m(z,\omega)c_r^*(z,\omega')} \rangle$ and $\langle \overline{c_m(z,\omega)c_n^*(z,\omega')} \; \overline{c_r(z,\omega'')c_k^*(z,\omega''')} \rangle$, where the c_m's are the slowly varying mode-amplitudes and the bar denotes the averaging operation over the source fluctuations. The knowledge of these quantities allows us in turn to describe the behaviour of the field and of its fluctuations in the general situation in which a finite-bandwidth carrier is amplitude or frequency-modulated.

An other significant result concerns the asymptotic behavior of the single-mode power fluctuations, which tend to vanish as a consequence of the finite carrier-bandwidth, so that, in the lossless case, power equipartition among modes takes place not only on the average, but over each fiber of the ensemble (5).

2. DESCRIPTION OF ELECTROMAGNETIC PROPAGATION IN A FIBER

The transverse part of the electric and magnetic fields excited in a fiber by means of a finite-bandwidth source can be approximately written in terms of the forward traveling guided modes $\underline{E}_m$, $\underline{H}_m$ as (1)

$$\underline{E}(\underline{r},z,t) = \sum_m \underline{E}_m(\underline{r},\omega_0) \int c_m(z,\omega) \, e^{i\omega t - i\beta_m(\omega)z} \, d\omega \; , \tag{1}$$

$$\underline{H}(\underline{r},z,t) = \sum_m \underline{H}_m(\underline{r},\omega_0) \int c_m(z,\omega) \, e^{i\omega t - i\beta_m(\omega)z} \, d\omega \; , \tag{2}$$

where $\underline{r}$ and z are the transverse and longitudinal coordinates, $\beta_m(\omega)$ represents the propagation constant of the mth mode at the (angular) frequency ω, and the realistic relation between the bandwidth $\delta\omega$ and the central frequency ω_0 of the signal

$$\delta\omega/\omega_0 \ll 1 \tag{3}$$

allows one to evaluate the mode configurations $\underline{E}_m(\underline{r},\omega)$ and $\underline{H}_m(\underline{r},\omega)$ at ω_0.

dence of the slowly-varying mode amplitudes c_m is associated with the departure of the fi_ber structure from the ideal one to which the configurations $\underline{E}_m$, $\underline{H}_m$ pertain. The electro_magnetic power carried through an arbitrary portion of area σ of any given fiber section z = const. can be expressed in terms of the complex Poynting vector $\underline{S}$ (6)

$$\underline{S} = \frac{1}{2}\,\overline{\underline{E} \times \underline{H}^*} \quad , \tag{4}$$

where the bar indicates the averaging operation over the source fluctuations, as

$$P^\sigma = \mathrm{Re} \int_\sigma \underline{e}_z \cdot \underline{S} \ d\underline{r} \quad . \tag{5}$$

Here "Re" means "real part of" and $\underline{e}_z$ is the unit vector in the positive z-direc_tion. By means of Eqs.(1),(2) and (4), Eq.(5) yields

$$P^\sigma(z,t) = \frac{1}{2} \sum_m \sum_r A_{mr}^\sigma(z,t) \quad , \tag{6}$$

where

$$A_{mr}^\sigma = \int_\sigma d\underline{r}\ \underline{E}_m(\underline{r},\omega_o) \times \underline{H}_r^*(\underline{r},\omega_o) \cdot \underline{e}_z \int d\omega \int d\omega'\ e^{i(\omega-\omega')t - i\left[\beta_m(\omega) - \beta_r(\omega')\right]z}\ \overline{c_m(z,\omega)c_r^*(z,\omega')}. \tag{7}$$

The total power P_t carried by the field is obtained by integrating $\underline{e} \cdot \underline{S}$ in Eq.(5) over the whole plane z = const. . This operation furnishes, with the help of the ortho_gonality relation between the modes (1)

$$\int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy\ \underline{e}_z \cdot \underline{E}_m(\underline{r},\omega_o) \times \underline{H}_n^*(\underline{r},\omega_o) = 2\,P\,\delta_{mn} \quad , \tag{8}$$

where P is a positive normalization constant and δ_{nm} is the usual Kronecker symbol,

$$P_t(z,t) = \sum_m P_m(z,t) \quad , \tag{9}$$

where

$$P_m(z,t) = P \int d\omega \int d\omega'\ e^{i(\omega-\omega')t - i\left[\beta_m(\omega) - \beta_m(\omega')\right]z}\ \overline{c_m(z,\omega)c_m^*(z,\omega')} \tag{10}$$

can be interpreted as the power carried by the mth mode. It is worthwhile to note that the interference terms between the various modes disappear in the expression of the to_tal power P_t, while they are present in that of P^σ.

3.   STATISTICAL DESCRIPTION OF PROPAGATION

The random nature of the imperfections unavoidably present in a real fiber suggests some statistical procedure as the most natural way for describing the propagation in opti_cal fibers. This is accomplished by introducing an ensemble of similar fibers, each of which differs from a common ideal model for the presence of random imperfections, and to evaluate the average of the significant quantities, e.g. P^σ, P and P_m, over this ensem_ble, which in turn is equivalent to consider $\langle \overline{c_m(z,\omega)c_r^*(z,\omega')} \rangle$ as the basic quantity.

In order to test the relevance of this approach for what concerns practical situa_tions, in which a single fiber is usually employed, one has to investigate the statisti_cal fluctuations around the average values. This leads to consider higher order averages of the kind $\langle \overline{c_m(z,\omega)c_r^*(z,\omega')}\ \overline{c_m(z,\omega'')c_r^*(z,\omega''')} \rangle, \langle \overline{c_m(z,\omega)c_r^*(z,\omega')}\ \overline{c_r(z,\omega'')c_m^*(z,\omega''')} \rangle$.

One starts from the system of equations describing the evolution of the c_m's in the single fiber, which reads (1)

$$\frac{dc_m(z,\omega)}{dz} = \sum_k A_{mk}(z,\omega)\ c_k(z,\omega) \quad , \tag{11}$$

with

$$A_{mk}(z,\omega) = K_{mk}(z)\, e^{i\left[\beta_m(\omega) - \beta_k(\omega)\right]z} \quad , \tag{12}$$

the K_{mk}'s being suitable coupling coefficients, vanishing in the case of an ideal fiber. By using Eq.(11) one can easily derive the following equations

$$\frac{d}{dz}\left[\overline{c_m^*(z,\omega')c_n(z,\omega)}\right] = \sum_k \left\{ A_{nk}(z,\omega)\, \overline{c_m^*(z,\omega')c_k(z,\omega)} \right.$$

$$\left. + A_{mk}^*(z,\omega')\, \overline{c_n(z,\omega)c_k^*(z,\omega')} \right\} \quad , \tag{13}$$

$$\frac{d}{dz}\left[\overline{c_m(z,\omega)c_r^*(z,\omega')}\ \overline{c_p(z,\omega'')c_n^*(z,\omega''')}\right] =$$

$$\sum_k \left\{ A_{mk}(z,\omega)\, \overline{c_r^*(z,\omega')c_k(z,\omega)}\ \overline{c_p(z,\omega'')c_n^*(z,\omega''')} \right.$$

$$+ \ A_{rk}^*(z,\omega')\, \overline{c_m(z,\omega)c_k^*(z,\omega')}\ \overline{c_p(z,\omega'')c_n^*(z,\omega''')}$$

$$+ \ A_{pk}(z,\omega'')\, \overline{c_n^*(z,\omega''')c_k(z,\omega'')}\ \overline{c_m(z,\omega)c_r^*(z,\omega')}$$

$$\left. + \ A_{nk}^*(z,\omega''')\, \overline{c_p(z,\omega'')c_k^*(z,\omega''')}\ \overline{c_m(z,\omega)c_r^*(z,\omega')} \right\} \quad , \tag{14}$$

which have to be averaged over the fiber ensemble. The resulting equations can be put in the form of closed systems under the homogeneity assumption

$$\langle K_{mn}(z) \rangle = 0 \quad , \tag{15}$$

$$\langle K_{mn}(z)K_{rp}(z-\zeta) \rangle = F_{mnrp}\left(|\zeta|\right) \quad , \tag{16}$$

provided that the coupling is small enough that the c_m's do not significantly vary over the correlation length of the statistical variables K_{mn}^m. In this way one obtains (see also Ref.(7)), as a first significant result,

$$\frac{d}{dz}\langle \overline{c_m(z,\omega)c_r^*(z,\omega')} \rangle = -\varepsilon_{mr}\langle \overline{c_m(z,\omega)c_r^*(z,\omega')} \rangle \quad , \ m \neq r \quad , \tag{17}$$

$$\frac{d}{dz}\langle \overline{c_m(z,\omega)c_r^*(z,\omega')}\ \overline{c_m(z,\omega'')c_r^*(z,\omega''')} \rangle =$$

$$-(2\varepsilon_{mr} + f_{mr})\langle \overline{c_m(z,\omega)c_r^*(z,\omega')}\ \overline{c_m(z,\omega'')c_r^*(z,\omega''')} \rangle \quad , \ m \neq r \quad , \tag{18}$$

with

$$\varepsilon_{mr} = \sum_k \int_0^{+\infty} \langle K_{mk}(\zeta)K_{km}(0) \rangle\, e^{-i\left[\beta_k(\omega_o) - \beta_m(\omega_o)\right]\zeta}\,d\zeta$$

$$+ \sum_k \int_0^{+\infty} \langle K_{rk}^*(\zeta)K_{kr}^*(0) \rangle\, e^{-i\left[\beta_r(\omega_o) - \beta_k(\omega_o)\right]\zeta}\,d\zeta$$

$$+ \int_{-\infty}^{+\infty} \langle K_{mm}(\zeta)K_{rr}^*(0) \rangle\,d\zeta \tag{19}$$

and

$$f_{mr} = \frac{1}{2}\lim_{L \to +\infty}\langle \left| (1/L)^{\frac{1}{2}}\int_{-L/2}^{L/2}\left[K_{mm}(z) - K_{rr}(z)\right]dz \right|^2 \rangle \quad . \tag{20}$$

The real part of ε_{mr} can be shown to be a positive quantity (7), so that

24.

$$\langle \overline{c_m(z,\omega)c_r^*(z,\omega')} \rangle = \overline{c_m(0,\omega)c_r^*(0,\omega')} \, e^{-g_{mr}z} \quad , \quad m \neq r , \tag{21}$$

and

$$\langle \overline{c_m(z,\omega)c_r^*(z,\omega')} \; \overline{c_m(z,\omega'')c_r^*(z,\omega''')} \rangle =$$

$$\overline{c_m(0,\omega)c_r^*(0,\omega')} \; \overline{c_m(0,\omega'')c_r^*(0,\omega''')} \, e^{-2g_{mr}z - f_{mr}z} \quad , \quad m \neq r , \tag{22}$$

vanish for z large enough (or are identically zero if they are such at z=0).

The lowest-order equations, concerning single-mode powers, read (see also Refs.(3) and (4))

$$\frac{d}{dz} \langle \overline{c_m(z,\omega)c_m^*(z,\omega')} \rangle = -\left(\sum_{k \neq m} h_{mk} \right) \langle \overline{c_m(z,\omega)c_m^*(z,\omega')} \rangle$$

$$+ \sum_{k \neq m} h_{mk} e^{i[\beta_m(\omega) - \beta_m(\omega') + \beta_k(\omega') - \beta_k(\omega)]z} \langle \overline{c_k(z,\omega)c_k^*(z,\omega')} \rangle \quad , \tag{23}$$

with

$$h_{mk} = \int_{-\infty}^{+\infty} \langle K_{mk}(\zeta)K_{mk}^*(0) \rangle \, e^{i[\beta_k(\omega_o) - \beta_m(\omega_o)]\zeta} d\zeta \quad , \tag{24}$$

which constitutes a closed set of equations.

The higher-order terms, concerning mode-power fluctuations and mode-mode correla_ tions, cannot be considered separately since they form together a closed set of equations, which reads

$$\frac{d}{dz} \langle \overline{c_m(z,\omega)c_m^*(z,\omega')} \; \overline{c_n(z,\omega'')c_n^*(z,\omega''')} \rangle =$$

$$-\left[\sum_k (h_{km} + h_{kn}) \right] \langle \overline{c_m(z,\omega)c_m^*(z,\omega')} \; \overline{c_n(z,\omega'')c_n^*(z,\omega''')} \rangle$$

$$+ \sum_k h_{mk} e^{i[\beta_m(\omega) - \beta_m(\omega') - \beta_k(\omega) + \beta_k(\omega')]z} \langle \overline{c_k(z,\omega)c_k^*(z,\omega')} \; \overline{c_n(z,\omega'')c_n^*(z,\omega''')} \rangle$$

$$+ \sum_k h_{nk} e^{i[\beta_n(\omega'') - \beta_n(\omega''') - \beta_k(\omega'') + \beta_k(\omega''')]z} \langle \overline{c_k(z,\omega'')c_k^*(z,\omega''')} \; \overline{c_m(z,\omega)c_m^*(z,\omega')} \rangle$$

$$-(1 - \delta_{nm})h_{mn}\left[e^{i[\beta_m(\omega) - \beta_m(\omega'') + \beta_n(\omega'') - \beta_n(\omega)]z} \langle \overline{c_n(z,\omega)c_m^*(z,\omega')} \; \overline{c_m(z,\omega'')c_n^*(z,\omega''')} \rangle \right.$$

$$+ e^{i[\beta_m(\omega''') - \beta_m(\omega') + \beta_n(\omega') - \beta_n(\omega''')]z} \langle \overline{c_m(z,\omega)c_n^*(z,\omega')} \; \overline{c_n(z,\omega'')c_m^*(z,\omega''')} \rangle \Big]$$

$$+ \delta_{nm}\left\{ \sum_{k \neq m} \left[h_{mk} e^{i[\beta_m(\omega) - \beta_k(\omega) + \beta_k(\omega''') - \beta_m(\omega''')]z} \langle \overline{c_k(z,\omega)c_m^*(z,\omega')} \; \overline{c_m(z,\omega'')c_k^*(z,\omega''')} \rangle \right. \right.$$

$$\left. \left. + h_{mk} e^{i[\beta_m(\omega'') - \beta_k(\omega'') + \beta_k(\omega') - \beta_m(\omega')]z} \langle \overline{c_m(z,\omega)c_k^*(z,\omega')} \; \overline{c_k(z,\omega'')c_m^*(z,\omega''')} \rangle \right] \right\} \tag{25}$$

and

$$\frac{d}{dz} \langle \overline{c_m(z,\omega)c_r^*(z,\omega')} \; \overline{c_r(z,\omega'')c_m^*(z,\omega''')} \rangle =$$

$$-\left(\sum_{k \neq r} h_{kr} + \sum_{k \neq m} h_{km} \right) \langle \overline{c_m(z,\omega)c_r^*(z,\omega')} \; \overline{c_r(z,\omega'')c_m^*(z,\omega''')} \rangle$$

$$+ \sum_{k \neq r} h_{rk} e^{i[\beta_r(\omega''') - \beta_r(\omega') + \beta_k(\omega') - \beta_k(\omega''')]z} \overline{\langle c_m(z,\omega) c_k^*(z,\omega') } \overline{ c_k(z;\omega''') c_m^*(z,\omega''') \rangle}$$

$$+ \sum_{k \neq m} h_{mk} e^{i[\beta_m(\omega) - \beta_m(\omega''') + \beta_k(\omega''') - \beta_k(\omega)]z} \overline{\langle c_k(z,\omega) c_r^*(z,\omega') } \overline{ c_r(z,\omega''') c_k^*(z,\omega''') \rangle}$$

$$- h_{mr} e^{i[\beta_m(\omega) - \beta_m(\omega''') + \beta_r(\omega''') - \beta_r(\omega)]z} \overline{\langle c_r(z,\omega) c_r^*(z,\omega') } \overline{ c_m(z,\omega''') c_m^*(z,\omega''') \rangle}$$

$$- h_{mr} e^{i[\beta_r(\omega''') - \beta_r(\omega') + \beta_m(\omega') - \beta_m(\omega''')]z} \overline{\langle c_m(z,\omega) c_m^*(z,\omega') } \overline{ c_r(z,\omega''') c_r^*(z,\omega''') \rangle} , \; m \neq r. \tag{26}$$

4.   SOME APPLICATIONS

The set of equations governing the behavior of the average mode-power $\langle P_m(z,t) \rangle$ can be easily obtained by taking advantage of Eqs.(10) and (23), thus getting (3),(4),(7)

$$\frac{\partial}{\partial z} \langle P_m(z,t) \rangle + \frac{1}{V_m} \frac{\partial}{\partial t} \langle P_m(z,t) \rangle = \sum_k h_{mk} \left[\langle P_k(z,t) \rangle - \langle P_m(z,t) \rangle \right] , \tag{27}$$

where

$$V_m = \left(\frac{d\beta_m}{d\omega} \right)^{-1}_{\omega = \omega_0} \tag{28}$$

is the group velocity of the mth mode. On the other hand, Eqs.(6),(7) and (10) allow one to write

$$\langle P^\sigma(z,t) \rangle = \sum_m \langle P_m^\sigma(z,t) \rangle + \frac{1}{2} \sum_{m \neq r} \sum \langle A_{mr}^\sigma \rangle , \tag{29}$$

where the $\langle P_m^\sigma(z,t) \rangle$ possess the same space-time dependence as the $\langle P_m(z,t) \rangle$. Since a frequency modulation of the signal does not affect the $\langle P_m \rangle$'s, the dependence on this modulation of $\langle P^\sigma(z,t) \rangle$ is contained in the $\langle A_{mr}^\sigma \rangle$'s, which vanish when considering the power transmitted through the whole section of the fiber. From Eqs.(7) and (17) it follows that (7)

$$\langle A_{mr}^\sigma(z,t) \rangle \propto e^{-g_{mr} z} \overline{\underline{P}(t - z/V_m) \underline{P}^*(t - z/V_r)} , \; m \neq r , \tag{30}$$

where

$$\underline{P}(t) = \underline{E}(z=0,t) e^{-i\omega_0 t} . \tag{31}$$

Equation (30) shows that mode-coupling affects the propagation of a frequency-modulated signal in an ensemble of fibers only for the presence of a spatial damping factor characteristic of each couple of modes. More precisely, if one excites at the fiber input only two modes m and r, all interference terms but $\langle A_{mr}^\sigma \rangle$ vanish throughout the fiber (see Eq.(17)), while the surviving term propagates as in the absence of coupling but for the spatial attenuation.

The set of Eqs.(23) and the set of Eqs.(25) and (26) allow one in principle to determine the quantities $\langle P_m(z,t) \rangle$, $\langle P^2(z,t) \rangle$ and $\langle P_m(z,t) P_r(z,t) \rangle$, and thus the fluctuations whose magnitude furnishes a criterion for the applicability of the statistical theory to a single fiber. While the general solution of Eqs.(25) and (26) is a formidable task, an important particular result can be obtained. If one introduces the energy $I_m(z)$ pertaining to the mtr mode

$$I_m(z) = \int_{-\infty}^{+\infty} P_m(z,t) dt = 2\pi P \int \overline{c_m(z,\omega) c_m^*(z,\omega)} \, d\omega , \tag{32}$$

it can be shown (7), by means of Eqs.(25), that, under the condition

$$z \gg \left| \left(\frac{1}{V_m} - \frac{1}{V_n} \right)^{-1} \right| \frac{2\pi}{\delta\omega} \quad , \, m \neq n \, , \tag{33}$$

and provided that z exceeds the equipartition distance beyond which $\langle I_m \rangle = \langle I_n \rangle$, the following relation holds

$$\frac{\langle I_m^2 \rangle - \langle I_m \rangle^2}{\langle I_m \rangle^2} \ll 1 \quad , \tag{34}$$

which implies that the statistical uncertainty of the quantity $I_m(z)$ over the fiber ensemble is negligible, so that mode-energy equipartition takes place over each fiber. This last consideration holds true in the absence of losses or, more in general, if losses are not effective over the characteristic length after which Eq.(34) is verified.

REFERENCES

1) D. Marcuse, "Theory of Dielectric Optical Waveguides", Academic Press, N.Y. 1974.

2) B. Crosignani, B. Daino, and P. Di Porto, IEEE Trans.Microwave Theory Tech. MTT-23, 416 (1975).

3) S.D. Personick, Bell Syst.Tech.J. 54, 47 (1975).

4) R. Steinberg, IEEE Trans.Microwave Theory Tech. MTT-23, 121 (1975).

5) B. Crosignani, B. Daino, and P. Di Porto, Opt Commun. 18, 551 (1976).

6) C.H. Papas, "Theory of Electromagnetic Wave Propagation", McGraw-Hill, N.Y. 1965.

7) B. Crosignani and C.H. Papas, Caltech Antenna Laboratory Report No. 76, 1976.